CONTRIBUTIONS TO THE MODEL THEORY OF LATTICE-ORDERED ALGEBRAS

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This thesis deals with the model theory of two kinds of lattice-ordered algebraic structures arising naturally in real algebraic geometry, both of which are understood within the context of real closed rings in the sense of Niels Schwartz.

The first such structure is the set $C_{\text{s.a.}}(X)$ of continuous semi-algebraic functions on a semi-algebraic curve $X \subseteq R^m$ over real closed field R regarded as a lattice-ordered module over itself. The model-theoretic analysis of $C_{\text{s.a.}}(X)$ is carried by an adaptation of the two-sorted machinery developed by Fuxing Shen and Volker Weispfenning in their study of first-order properties of lattice-ordered abelian groups of functions in terms of their lattices of zero sets. The lattice-ordered module $C_{\text{s.a.}}(X)$ is enriched with a sort for its lattice of zero sets L_X (the space sort), a sort for a real closed valuation ring $\mathcal{O}_R$ (the germ sort), and suitable maps connecting these sorts. It is shown that the resulting three-sorted structure eliminates quantifiers relative to the space and germ sorts, from which it follows that every first-order property of the lattice-ordered module $C_{\text{s.a.}}(X)$ is equivalent to a Boolean combination of first-order properties of L_X and $\mathcal{O}_R$. Under the additional hypothesis that R is a recursive real closed field, this equivalence is effective, from which decidability of the theory of the lattice-ordered module $C_{\text{s.a.}}(X)$ is obtained.

The second class of structures is that of n-fold fibre products of non-trivial real closed valuation rings along surjective maps onto a fixed domain D. For a fixed $n \geq 2$, this class splits into two subclasses, called in this thesis rings of type $(n, 1)$ and of type $(n, 2)$ according to whether D is a field or not (respectively). Geometric examples of rings of type $(n, 1)$ are rings of germs of $C_{\text{s.a.}}(X)$ at a point $a \in X$, where X is a semi-algebraic curve as above. It is shown that these two classes admit a simple axiomatization in the language of rings and their basic model-theoretic properties are established, namely, completeness, decidability, and NIP, as well as model completeness and quantifier elimination in suitable enrichments of the language of rings. These model-theoretic results rest on a structure theorem proved for reduced local SV-rings of finite rank which explicitly describes them as finite iterated fibre products of non-trivial valuation rings along surjective ring homomorphisms, as well as on equivalent descriptions of branching ideals in local real closed rings of finite rank. The algebraic and model-theoretic study of of rings of type $(n, 1)$ and of type $(n, 2)$ is framed within the larger class of local real closed SV-rings of finite rank, and the results obtained in this thesis pave the way toward a uniform model-theoretic treatment of these latter rings in terms of their branching spectra.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

A lattice is here understood to be a partially ordered set $(L, \leq)$ such that the supremum $a \vee b$ and the infimum $a \wedge b$ of every pair of elements $a, b \in L$ with respect to $\leq$ exist in L. Lattices are ubiquitous algebraic and ordered structures which appear in diverse areas of mathematics such as topology, algebra, and logic.

This thesis contributes to the model-theoretic analysis (that is, the study of first-order logical properties) of certain algebraic structures arising in real algebraic geometry which carry a lattice order compatible with their underlying algebraic operations, namely:

- The lattice-ordered module $C_{\text{s.a.}}(X)$ (over itself) of continuous semi-algebraic functions $X \longrightarrow R$ on a one-dimensional semi-algebraic subset $X \subseteq R^m$ without isolated points over a real closed field R. This is the content of Chapter 3.
- Local real closed SV-rings of finite rank; equivalently (Theorem 4.4.2), finite iterated fibre products of non-trivial real closed valuation rings along surjective ring homomorphisms onto domains. This is the content of Chapter 4.

$C_{\text{s.a.}}(X)$ can also be regarded as a ring with pointwise addition and multiplication of functions. In particular, a first connection between the content of Chapters 3 and 4 is that both the ring $C_{\text{s.a.}}(X)$ and local real closed SV-rings of finite rank are particular examples of real closed rings in the sense of Niels Schwartz. Real closed rings are a class of lattice-ordered rings introduced by Schwartz in [Sch89] (see also [Sch87]) to serve as rings of global sections of affine real closed spaces, the latter being analogues of Grothendieck's affine schemes in the context of real algebraic geometry.

A second and less obvious point in common between Chapters 3 and 4 is that the ring of germs of functions $f \in C_{\text{s.a.}}(X)$ at a point $a \in X$ (that is, the localization of the ring $C_{\text{s.a.}}(X)$ the maximal ideal $\mathfrak{m}_a := \{f \in C_{\text{s.a.}}(X) \mid f(a) = 0\}$) is a local real closed SV-ring of finite rank. More precisely, if n is the number of half-branches of the curve X at a, then the ring of germs $C_{\text{s.a.}}(X)_{\mathfrak{m}_a}$ is isomorphic to the n-fold fibre product $(((\mathcal{O}_R \times_R \mathcal{O}_R) \times_R \dots) \times_R \mathcal{O}_R)$, where $\mathcal{O}_R$ is the ring of germs of function $f \in C_{\text{s.a.}}(X)$ at a half-branch of X, see Corollary 2.3.37 and Example 4.4.9.

The third and last common feature of Chapters 3 and 4 is that both use the well known model-theoretic properties of real closed valuation rings established by Cherlin and Dickmann in [CD83] for the analysis of the first-order properties of the structures dealt with in each chapter. That such analysis can be done in this way is witnessed algebraically by the fact that both the ring $C_{\text{s.a.}}(X)$ and local real closed SV-rings of finite rank admit a sheaf representation on a spectral space whose stalks are real closed valuation rings, see [Sch91].

The model-theoretic analysis of the lattice-ordered structures dealt with in this thesis is motivated by the following:

Question. Let $S \subseteq \mathbb{R}^m$ be a semi-algebraic subset. Does the real closed ring $C_{\text{s.a.}}(S)$ of continuous semi-algebraic functions $S \longrightarrow \mathbb{R}$ have a decidable first-order theory when S is of dimension 1?

The question above arises from the decidability and non-decidability results in the literature of various theories $\text{Th}(A)$ in the language of rings of real closed rings A of functions $S \longrightarrow \mathbb{R}$, namely:

(i) $\text{Th}(A)$ is decidable if A is the ring of all functions $S \longrightarrow \mathbb{R}$ by the Feferman-Vaught theorem in [FV59].

(ii) $\text{Th}(A)$ is decidable if A is the ring of all semi-algebraic functions $S \longrightarrow \mathbb{R}$ by Astier's [Ast08, Theorem 1].

(iii) $\text{Th}(A)$ is undecidable if S has non-empty interior and A is the ring of all continuous functions $S \longrightarrow \mathbb{R}$ by Cherlin's [Che80, Theorem I].

(iv) $\text{Th}(A)$ is undecidable if S is semi-algebraically connected of constant local dimension at least 2 and A is the ring of all continuous semi-algebraic functions

$S \longrightarrow \mathbb{R}$ by [DT20, Theorem 6.7].

The results obtained in this thesis contribute to the above question by showing that if R is a real closed field and $X \subseteq R^m$ is semi-algebraic of dimension 1, then

(a) If R is a recursive real closed field, then the theory of the lattice-ordered module $C_{\text{s.a.}}(X)$ over itself is decidable, see Proposition 3.5.3 (ii).

(b) The theory of the ring of germs of functions $f \in C_{\text{s.a.}}(X)$ at any point $a \in X$ is decidable, see Example 4.4.9 and Corollary 4.5.28.

1.1 Structure of the thesis

Chapter 2 contains all background material needed for Chapters 3 and 4. Most of the results in Chapter 2 are required preliminaries for Chapter 3, while only the content in Section 2.3 is a required preliminary for Chapter 4. In particular, the essential background material for Chapter 3 is Subsection 2.4.3 on the Shen-Weispfenning theorem for lattice-ordered abelian groups of functions.

Chapters 3 and 4 can be read independently. The introductions in Sections 3.1 and 4.1 explain the algebraic and model-theoretic context and set-up of the contents of Chapters 3 and 4, respectively, as well as the structure within each chapter and a summary of the main results obtained.

1.2 General conventions and notation

(I) All rings are commutative and unital, and all ring homomorphisms preserve the multiplicative unit. The category of rings together with ring homomorphisms is denoted by **CRing**.

(II) A *partially ordered ring* (*poring* for short) is a ring A equipped with a partial order $\leq$ such that for all $a, b, c \in A$, if $a \leq b$ then $a + c \leq b + c$, and if $0 \leq a, b$ then $0 \leq ab$.

(III) If A is a domain, then $\mathrm{qf}(A)$ is the quotient field of A.

(IV) If A is a partially ordered group, set $A^{\geq 0} := \{a \in A \mid a \geq 0\}$.

(V) If $f : X \longrightarrow N$ is any function, then $\mathrm{im}(f)$ is its image and $f_{\restriction S}$ the restriction of f to $S \subseteq X$.

(VI) $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \,\dot{\cup}\, \{0\} = \omega$.

(VII) If $n \in \mathbb{N}$, then $[n] := \{1, \ldots, n\}$.

(VIII) If X is any set, then $\mathcal{P}(X)$ is the power set of X.

(IX) $\bigvee$ and $\bigwedge$ denote logical disjunction and conjunction, respectively.

1.2.1 Prerequisites

In this thesis it is assumed familiarity with the following areas of mathematics:

- First-order model theory ([Mar02], [CK90], and [Hod93]).
- Semi-algebraic geometry and o-minimality ([BCR98], [Dri98], and [PS86]).
- Commutative algebra ([AM69] and [MR89]).

Some working knowledge of basic categorical notions such as pullback or functor is also assumed; for such notions a standard reference is [Mac98]. The level of depth assumed in the areas above is that of an introductory graduate-level course. The reader is referred to the aforementioned sources for details on any notion corresponding to one of the these areas which appears undefined in this thesis, such as model completeness, semi-algebraic function, or prime ideal.

Chapter 2

Preliminaries

The central classes of lattice-ordered algebras that are dealt with in this thesis are real closed rings and lattice-ordered abelian groups. As such, one may divide the four sections of this chapter in:

- Main sections: real closed rings (Section 2.3) and lattice-ordered abelian groups (Section 2.4).
- Secondary sections: model theory (Section 2.1) and spectral spaces (Section 2.2).

Sections 2.3 and 2.4 can be read independently of each other, and these two main sections both make use of part of the theory presented in the secondary sections. The extent to which Sections 2.1 and 2.2 are secondary will be clear from the outline of the content of this chapter that now follows.

Section 2.1 on model theory deals with two separate topics. The first one is that of relative quantifier elimination. This syntactic property of multi-sorted theories is the key notion needed to formalize two of the main theorems in this thesis, namely, the Shen-Weispfenning theorem on lattice-ordered groups of functions (Subsection 2.4.3), and the main theorem of Chapter 3, whose proof builds on the ideas behind the proof of the Shen-Weispfenning theorem. The second topic is that of recursive model theory, which deals with the effective content of model-theoretic constructions. This is included in order to set-up the right framework to obtain decidability results from the two aforementioned theorems. Here an important distinction is made between the theory of a structure being decidable and the structure itself being decidable, the latter being a crucial ingredient for the decidability results in Chapter 3.

The role that spectral spaces play in this thesis is that of being spaces X on which both real closed rings and lattice-ordered abelian groups can be represented as algebras of functions $X \longrightarrow N$ for a suitable choice of totally ordered structure N. Section 2.2 introduces all the required notation and terminology on spectral spaces that is needed to use such functional representations in order to apply the Shen-Weispfenning theorem to the additive group reduct of a real closed ring and to an arbitrary divisible lattice-ordered abelian group.

Section 2.3 contains all the relevant background on real closed rings needed for the development of both Chapters 3 and 4. Here a particular emphasis is made on real closed valuation rings for two reasons: a particular real closed valuation ring is used to obtain the relative quantifier elimination statement of Chapter 3 (namely, the ring of germs of continuous semi-algebraic functions on a semi-algebraic curve at a half-branch, see Subsection 2.3.2), and real closed valuation rings are used to construct the class of real closed rings which is dealt with in Chapter 4.

Lattice-ordered abelian groups are introduced in Section 2.4 just to the extent needed to contextualize and prove the Shen-Weispfenning theorem in Subsection 2.4.3. The core of this section is in fact the content in Subsection 2.4.3, since it serves as a basic template for the set-up and proof of the main theorem in Chapter 3.

Most of the material in this chapter is folklore and well known. Those results which do not appear in the literature or which are of difficult access include a proof.

2.1 Model theory

Fix the following conventions, notation, and terminology which will be used throughout this section:

(I) Every language $\mathscr{L}$ is a (possibly multi-sorted) first-order language.

(II) Constant symbols $c \in \mathscr{L}$ are regarded as 0-ary function symbols.

(III) The interpretation in an $\mathscr{L}$-structure $\mathscr{M}$ of a non-logical symbol or a sort in $\mathscr{L}$ will be indicated with the superscript notation $(-)^{\mathscr{M}}$ whenever this is needed.

(IV) If $\mathscr{L}$ is a language, then $\mathscr{L}$-Fml is the set of all $\mathscr{L}$-formulas.

(V) An $\mathscr{L}$-*theory* is[1] a consistent and deductively closed set of $\mathscr{L}$-sentences.

(VI) If $\mathscr{M}$ is an $\mathscr{L}$-structure, then an $\mathscr{L}$-formula $\varphi(\overline{x})$ is *equivalent to* $\psi(\overline{x})$ *modulo* $\mathscr{M}$ if $\mathscr{M} \models \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$. If T is an $\mathscr{L}$-theory, then an $\mathscr{L}$-formula $\varphi(\overline{x})$ is *equivalent to* $\psi(\overline{x})$ *modulo* T if $\varphi(\overline{x})$ is equivalent to $\psi(\overline{x})$ modulo every $\mathscr{M} \models T$; equivalently, $T \vdash \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$.

2.1.1 Relative quantifier elimination

Familiarly with basic many-sorted model theory is assumed for this subsection (see for instance Appendix B in [ADH17]), but all the relevant notions will be recalled as required. A simple example of a two-sorted structure to have in mind for this subsection and which will be particularly relevant for Section 2.4 and Chapter 3 is the following: the group $\mathbb{R}^{\mathbb{R}}$ of all functions $\mathbb{R} \longrightarrow \mathbb{R}$ under pointwise addition together with the "valuation-like" map $\mathbb{R}^{\mathbb{R}} \longrightarrow \mathcal{P}(\mathbb{R})$ to the Boolean algebra $\mathcal{P}(\mathbb{R})$ given by $f \mapsto \{x \in \mathbb{R} \mid f(x) = 0\}$, where one sort is the group $\mathbb{R}^{\mathbb{R}}$ and the other sort is the Boolean algebra $\mathcal{P}(\mathbb{R})$.

Much of the material of this subsection is contained in [Rid14, Chapter II, Appendix A].

Throughout this subsection, $\mathscr{L}$ is a multi-sorted language with partition of sorts $\Pi \dot\cup \Sigma$ and T is an $\mathscr{L}$-theory.

For what follows, recall that every function and relation symbol in the multi-sorted language $\mathscr{L}$ is equipped with a *sort* (cf. [ADH17, Appendix B1 - B4]): if $f \in \mathscr{L}$ is a function symbol, then the *sort of* f is a tuple of sorts $(S_1, \ldots, S_m, S_{m+1})$ such that $f^{\mathscr{M}}$ is a function $S_1^{\mathscr{M}} \times \ldots \times S_m^{\mathscr{M}} \longrightarrow S_{m+1}^{\mathscr{M}}$ for every $\mathscr{L}$-structure $\mathscr{M}$ (and thus f is in particular an m-ary function symbol), and the sort of a relation symbol is defined analogously. Similarly, every variable is equipped with a unique single sort, and given $\varphi(x_1, \ldots, x_n) \in \mathscr{L}$-Fml, the *sort of* $\varphi(x_1, \ldots, x_n)$ is the tuple of sorts $(S_1, \ldots, S_n)$, where S_i is the sort of x_i for all $i \in [n]$; for instance, if f is a function

[1] This definition of a theory strays away from the standard model-theoretic definition of a theory (that is, that an $\mathscr{L}$-theory is just a set of $\mathscr{L}$-sentences). The assumption that theories in this thesis are consistent and deductively closed is used when discussing matters of recursiveness and decidability; this is in accordance to the literature in recursive model theory, see [Rab77, Subsection 1.1] or [Har98, Section 2].

symbol of sort $(S_1, \ldots, S_m, S_{m+1})$, then the formula $f(x_1, \ldots, x_m) = x_{m+1}$ is of sort $(S_1, \ldots, S_m, S_{m+1})$.

Definition 2.1.1. (i) A function symbol in $\mathscr{L}$ is Σ-*sorted* (Π-*sorted*) if it is of sort $(S_1, \ldots, S_m, S_{m+1})$ with $S_i \in \Sigma$ ($S_i \in \Pi$) for all $i \in [m+1]$; a Σ-sorted relation symbol in $\mathscr{L}$ is defined analogously, and a variable x is a Σ-*variable* (Π-*variable*) if x is of sort S for some $S \in \Sigma$ ($S \in \Pi$). Write $\mathscr{L}_{\restriction\Sigma}$ ($\mathscr{L}_{\restriction\Pi}$) for the restriction of $\mathscr{L}$ to Σ-sorted (Π-sorted) function and relation symbols.

(ii) The *Morleyization of* $\mathscr{L}$ *on* Σ is the language

$$\mathscr{L}^{\Sigma\text{-Mor}} := \mathscr{L} \,\dot{\cup}\, \{R_\varphi(\overline{x}) \mid \varphi(\overline{x}) \in \mathscr{L}_{\restriction\Sigma}\text{-Fml}\},$$

where $R_\varphi(\overline{x})$ is a new relation symbol of sort $(S_1, \ldots, S_{|\overline{x}|})$ for each $\varphi(\overline{x}) \in \mathscr{L}_{\restriction\Sigma}$-Fml of sort $(S_1, \ldots, S_{|\overline{x}|})$.

(iii) The *Morleyization of* T *on* Σ is the $\mathscr{L}^{\Sigma\text{-Mor}}$-theory

$$T^{\Sigma\text{-Mor}} := T \,\dot{\cup}\, \{\forall \overline{x}(R_\varphi(\overline{x}) \leftrightarrow \varphi(\overline{x})) \mid \varphi(\overline{x}) \in \mathscr{L}_{\restriction\Sigma}\text{-Fml}\}.$$

(iv) T *eliminates quantifiers relative to* Σ if $T^{\Sigma\text{-Mor}}$ has quantifier elimination, that is, if every $\mathscr{L}^{\Sigma\text{-Mor}}$-formula is equivalent modulo $T^{\Sigma\text{-Mor}}$ to a formula without quantifiers; if $\Sigma = \{S\}$, then say that T *eliminates quantifiers relative to* S.

(v) T *eliminates* Π-*quantifiers* if every $\mathscr{L}$-formula is equivalent modulo T to a formula without Π-quantifiers; if $\Pi = \{S\}$, then say that T *eliminates* S-*quantifiers.*

Remark 2.1.2. If T eliminates quantifiers relative to Σ, then T eliminates Π-quantifiers. Indeed, let $\varphi(\overline{x}, \overline{y})$ be an $\mathscr{L}$-formula (where $\overline{x}$ are Π-variables and $\overline{y}$ are Σ-variables); then $\varphi(\overline{x}, \overline{y})$ is also an $\mathscr{L}^{\Sigma\text{-Mor}}$-formula, therefore by assumption there exists an $\mathscr{L}^{\Sigma\text{-Mor}}$-formula $\varphi_1(\overline{x}, \overline{y})$ without quantifiers which is equivalent to $\varphi(\overline{x}, \overline{y})$ modulo $T^{\Sigma\text{-Mor}}$. Let $\varphi_2(\overline{x}, \overline{y})$ be the $\mathscr{L}$-formula obtained by replacing every atomic $\mathscr{L}^{\Sigma\text{-Mor}}$-subformula of $\varphi_1(\overline{x}, \overline{y})$ of the form $R_\theta(\overline{z}, \overline{y})$ (where $\theta(\overline{z}, \overline{y})$ is an $\mathscr{L}_{\restriction\Sigma}$-formula) by $\theta(\overline{z}, \overline{y})$; then $\varphi_2(\overline{x}, \overline{y})$ is an $\mathscr{L}$-formula without Π-quantifiers which is equivalent to $\varphi(\overline{x}, \overline{y})$ modulo T.

Definition 2.1.3 (Definition II.A.7 in [Rid14])**.** The set of sorts Σ is *closed (in* $\mathscr{L}$*)* if $\mathscr{L} \setminus (\mathscr{L}_{\restriction\Pi} \,\dot{\cup}\, \mathscr{L}_{\restriction\Sigma})$ is either empty or it consists only of function symbols f of sort $(S_1, \ldots, S_m, S_{m+1})$ with $m \in \mathbb{N}$, $S_i \in \Pi$ for all $i \in [m]$, and $S_{m+1} \in \Sigma$. If Σ is closed, define $\mathcal{F}_\Sigma := \mathscr{L} \setminus (\mathscr{L}_{\restriction\Pi} \,\dot{\cup}\, \mathscr{L}_{\restriction\Sigma})$.

Loosely speaking, if Σ is closed, then the only interaction between the sorts in Π and the sorts in Σ is via function symbols in

$$\mathcal{F}_\Sigma = \{f \in \mathscr{L} \mid f \text{ is a function symbol of sort } (S_1, \ldots, S_m, S_{m+1}) \\ \text{with } m \in \mathbb{N},\ S_i \in \Pi \text{ for all } i \in [m],\ \text{and } S_{m+1} \in \Sigma\}.$$

Remark 2.1.4. Σ is closed if and only if the following conditions hold:

(i) if $f \in \mathscr{L}$ is a function symbol of sort $(S_1, \ldots, S_m, S_{m+1})$ with $m \in \mathbb{N}$ such that $S_{m+1} \in \Pi$, then $S_i \in \Pi$ for all $i \in [m]$;

(ii) if $f \in \mathscr{L}$ is a function symbol of sort $(S_1, \ldots, S_m, S_{m+1})$ with $m \in \mathbb{N}$ such that $S_{m+1} \in \Sigma$ and there exists $i \in [m]$ such that $S_i \in \Sigma$, then $S_i \in \Sigma$ for all $i \in [m]$; and

(iii) if $R \in \mathscr{L}$ is a relation symbol of sort $(S_1, \ldots, S_n)$ and there exists $i \in [n]$ such that $S_i \in \Sigma$, then $S_i \in \Sigma$ for all $i \in [n]$.

Remark 2.1.5. If Σ is closed, then any atomic $\mathscr{L}$-formula $\varphi(\overline{x}, \overline{y})$ (where $\overline{x}$ are Π-variables and $\overline{y}$ are Σ-variables) is of the form

(i) $\psi(\overline{x})$ for some atomic $\mathscr{L}_{\restriction\Pi}$-formula $\psi(\overline{x})$, or

(ii) $\theta(f_1(\overline{u_1}(\overline{x})), \ldots, f_r(\overline{u_r}(\overline{x})), \overline{y})$, where $\theta(z_1, \ldots, z_r, \overline{y})$ is an atomic $\mathscr{L}_{\restriction\Sigma}$-formula, and for all $i \in [r]$, $\overline{u_i}(\overline{x})$ is a tuple of $\mathscr{L}_{\restriction\Pi}$-terms and f_i is a function symbol in $\mathcal{F}_\Sigma$ of the appropriate sort.

Lemma 2.1.6. *Suppose that Σ is closed. If $\mathcal{F}_\Sigma = \varnothing$, then every $\mathscr{L}$-formula is equivalent to a Boolean combination of $\mathscr{L}_{\restriction\Pi}$-formulas and $\mathscr{L}_{\restriction\Sigma}$-formulas.*

Proof. By writing $\mathscr{L}$-formulas in prenex normal form[2] and inducting over quantifiers, it suffices to show that the $\mathscr{L}$-formula $\exists z \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \varphi_{i,j}(\overline{x}, \overline{y}, z)$ is equivalent to a Boolean combination of $\mathscr{L}_{\restriction\Pi}$-formulas and $\mathscr{L}_{\restriction\Sigma}$-formulas, where z is either a Π-variable or a Σ-variable, and each $\varphi_{ij}(\overline{x}, \overline{y}, z)$ is an atomic $\mathscr{L}$-formula such that $\overline{x}$ are Π-variables and $\overline{y}$ are Σ-variables. Assume without loss of generality that z is a Σ-variable; since Σ is closed, $\mathcal{F}_\Sigma = \varnothing$ together with Remark 2.1.5 imply that $\varphi_{ij}(\overline{x}, \overline{y}, z)$

[2]The proof that every formula in a multi-sorted language is equivalent to a formula in prenex normal form (i.e., a formula consisting of a possibly empty string of quantifiers followed by a quantifier-free formula) is analogous to the one-sorted case.

is either an atomic $\mathscr{L}_{\restriction\Pi}$-formula $\varphi_{ij}(\overline{x})$ or an atomic $\mathscr{L}_{\restriction\Sigma}$-formula $\varphi_{ij}(\overline{y}, z)$ for all $i \in [m]$ and for all $j \in [n]$, therefore it can be assumed that for each $i \in [m]$ there exists $n_i \in [n]$ such that $\varphi_{ij}(\overline{x}, \overline{y}, z)$ is an atomic $\mathscr{L}_{\restriction\Pi}$-formula $\varphi_{ij}(\overline{x})$ for all $j \in [n_i]$ and $\varphi_{ij}(\overline{x}, \overline{y}, z)$ is an atomic $\mathscr{L}_{\restriction\Sigma}$-formula $\varphi_{ij}(\overline{y}, z)$ for all $j \in \{n_i+1, \ldots, n\}$. It is clear that

$$\bigvee_{j=1}^{m} \left(\bigwedge_{i=1}^{n_i} \varphi_{ij}(\overline{x}) \wedge \left(\exists z \bigwedge_{i=n_i+1}^{n} \varphi_{ij}(\overline{y}, z) \right) \right)$$

is a formula equivalent to the original one and that it is in the required form. □

Lemma 2.1.7 (Remark II.A.8.3 in [Rid14]). *Suppose that* Σ *is closed. Then* T *eliminates* Π*-quantifiers if and only if* T *eliminates quantifiers relative to* Σ*.*

Proof. One implication follows by Remark 2.1.2, so suppose that T eliminates Π-quantifiers. By writing $\mathscr{L}^{\Sigma\text{-mor}}$-formulas in prenex normal form and inducting over quantifiers, it suffices to eliminate quantifiers in formulas of the form $\exists\overline{y}\varphi(\overline{x}, \overline{y})$ and $\exists\overline{x}\varphi(\overline{x}, \overline{y})$, where $\varphi(\overline{x}, \overline{y})$ is a quantifier-free $\mathscr{L}^{\Sigma\text{-Mor}}$-formula, $\overline{x}$ are Π-variables, and $\overline{y}$ are Σ-variables.

Since Σ is closed, it can be assumed by Remark 2.1.5 that each atomic subformula of $\varphi(\overline{x}, \overline{y})$ is of the form

(i) $\psi(\overline{x})$ for some atomic $\mathscr{L}_{\restriction\Pi}$-formula $\psi(\overline{x})$, or

(ii) $R_\theta(f_1(\overline{u_1}(\overline{x})), \ldots, f_r(\overline{u_r}(\overline{x})), \overline{y})$, where θ, f_i, and u_i are as in Remark 2.1.5 (ii);

in particular, it can be assumed that $\varphi(\overline{x}, \overline{y})$ is a finite conjunction of formulas of the form as in items (i) and (ii) above. Let $R_{\theta_1}(\overline{f}(\overline{u}(\overline{x})), \overline{y}), \ldots, R_{\theta_n}(\overline{f}(\overline{u}(\overline{x})), \overline{y})$ be a complete list of all atomic subformulas of $\varphi(\overline{x}, \overline{y})$ of the form as in item (ii) above, and let $\varphi'(\overline{x})$ be the $\mathscr{L}^{\Sigma\text{-Mor}}$-formula defined by replacing the subformula $\bigwedge_{i=1}^{n} R_{\theta_i}(\overline{f}(\overline{u}(\overline{x})), \overline{y})$ of $\varphi(\overline{x}, \overline{y})$ by $R_{\exists\overline{y} \bigwedge_{i=1}^{n} \theta_i}(\overline{f}(\overline{u}(\overline{x}))$; then $\varphi'(\overline{x})$ is a quantifier-free $\mathscr{L}^{\Sigma\text{-Mor}}$-formula equivalent to $\exists\overline{y}\varphi(\overline{x}, \overline{y})$ modulo $T^{\Sigma\text{-Mor}}$.

Let $\varphi_1(\overline{y})$ be an $\mathscr{L}$-formula equivalent to $\exists\overline{x}\varphi(\overline{x}, \overline{y})$ modulo $T^{\Sigma\text{-Mor}}$ (this exists since $T^{\Sigma\text{-Mor}}$ is an extension of T by definitions). By assumption, there exists an $\mathscr{L}$-formula $\varphi_2(\overline{y})$ without Π-quantifiers which is equivalent to $\varphi_1(\overline{y})$ modulo T, therefore $\varphi_2(\overline{y})$ is an $\mathscr{L}^{\Sigma\text{-Mor}}$-formula without Π-quantifiers equivalent to $\exists\overline{x}\varphi(\overline{x}, \overline{y})$ modulo $T^{\Sigma\text{-Mor}}$; by the above, $\varphi_2(\overline{y})$ is equivalent to a quantifier-free $\mathscr{L}^{\Sigma\text{-Mor}}$-formula modulo $T^{\Sigma\text{-Mor}}$, and this concludes the proof. □

This section concludes with a syntactic test for relative quantifier elimination, see Proposition 2.1.9. The proof idea is essentially a generalization of the proof of the main theorem in [SW87a].

Lemma 2.1.8. *Suppose that Σ is closed and that every atomic $\mathscr{L}_{\restriction\Pi}$-formula $\psi(\overline{x})$ is equivalent modulo T to an $\mathscr{L}$-formula of the form $\theta(f_1(\overline{u_1}(\overline{x})),\dots,f_r(\overline{u_r}(\overline{x})))$, where $\theta(z_1,\dots,z_r)$ is an atomic $\mathscr{L}_{\restriction\Sigma}$-formula, and for all $i\in[r]$, $\overline{u_i}(\overline{x})$ is a tuple of $\mathscr{L}_{\restriction\Pi}$-terms and f_i is a function symbol in $\mathcal{F}_\Sigma$ of the appropriate sort. Then every $\mathscr{L}$-formula $\varphi(\overline{x},\overline{y})$ without Π-quantifiers (where $\overline{x}$ are Π-variables and $\overline{y}$ are Σ-variables) is equivalent modulo T to a formula of the form*

$$\exists\overline{z}\left(\gamma(\overline{z},\overline{y}) \wedge\!\!\!\wedge \bigwedge\!\!\!\!\bigwedge_{i=1}^{m} z_i = f_i(\overline{u_i}(\overline{x}))\right), \tag{†}$$

where $\gamma(\overline{z},\overline{y})$ is an $\mathscr{L}_{\restriction\Sigma}$-formula, and for all $i\in[m]$, $\overline{u_i}(\overline{x})$ is a tuple of $\mathscr{L}_{\restriction\Pi}$-terms and f_i is a function symbol in $\mathcal{F}_\Sigma$ of the appropriate sort. Moreover, if $\varphi(\overline{x},\overline{y})$ is an existential formula[3]*, then the resulting equivalent formula in (†) is also existential.*

Proof. Let $\varphi(\overline{x},\overline{y})$ be an $\mathscr{L}$-formula without Π-quantifiers. By writing $\varphi(\overline{x},\overline{y})$ in prenex normal form, it may be assumed that $\varphi(\overline{x},\overline{y})$ is of the form

$$Q_1v_1\dots Q_nv_n\gamma_1(\overline{x},\overline{y},\overline{v}) \tag{2.1}$$

where $\gamma_1(\overline{x},\overline{y},\overline{v})$ is a Boolean combination of atomic $\mathscr{L}$-formulas, and for all $i\in[n]$, Q_i is either $\forall$ or $\exists$, and v_i is a Σ-variable. By Remark 2.1.5 and by assumption, (2.1) is equivalent modulo T to a formula of the form

$$Q_1v_1\dots Q_nv_n\gamma_2(\overline{x},\overline{y},\overline{v}) \tag{2.2}$$

where Q_i and v_i are as above for all $i\in[n]$, and $\gamma_2(\overline{x},\overline{y},\overline{v})$ is a Boolean combination of formulas of the form

$$\theta(f_1(\overline{u_1}(\overline{x})),\dots,f_r(\overline{u_r}(\overline{x})),\overline{y},\overline{v}),$$

where $\theta(z_1,\dots,z_r,\overline{y},\overline{v})$ is an atomic $\mathscr{L}_{\restriction\Sigma}$-formula, and for all $j\in[r]$, $\overline{u_j}(\overline{x})$ is a tuple of $\mathscr{L}_{\restriction\Pi}$-terms and f_j is a function symbol in $\mathcal{F}_\Sigma$ of the appropriate sort. Let

[3] Here an *existential formula* is one whose prenex normal form consists of a block of existential quantifiers followed by a quantifier-free formula.

$f_1(\overline{u}_1(\overline{x})), \ldots, f_m(\overline{u_m}(\overline{x}))$ be a complete list of all the terms appearing in γ_2 which are of the form $f(\overline{u}(\overline{x}))$, where $\overline{u}(\overline{x})$ is a tuple of $\mathscr{L}_{\restriction\Pi}$-terms and f is a function symbol in $\mathcal{F}_\Sigma$ of the appropriate sort, and let $\gamma_3(\overline{z}, \overline{y}, \overline{v})$ be the formula obtained by replacing each occurrence of the term $f_i(\overline{u}_i(\overline{x}))$ in γ_2 by a new Σ-variable z_i for all $i \in [m]$; then γ_3 is an $\mathscr{L}_{\restriction\Sigma}$-formula by construction, and (2.2) is equivalent modulo T to

$$\exists \overline{z} \left(\gamma(\overline{z}, \overline{y}) \wedge\!\!\!\wedge \bigwedge\!\!\!\!\bigwedge_{i=1}^{m} z_i = f_i(\overline{u_i}(\overline{x})) \right),$$

where $\gamma(\overline{z}, \overline{y})$ is defined as $Q_1 v_1 \ldots Q_n v_n \gamma_3(\overline{z}, \overline{y}, \overline{v})$. The moreover part in the statement of the lemma is clear by construction. □

Proposition 2.1.9. *Suppose that Σ is closed. Suppose further that:*

(i) *every atomic $\mathscr{L}_{\restriction\Pi}$-formula $\psi(\overline{x})$ is equivalent modulo T to an $\mathscr{L}$-formula of the form $\theta(f_1(\overline{u_1}(\overline{x})), \ldots, f_r(\overline{u_r}(\overline{x})))$, where $\theta(z_1, \ldots, z_r)$ is an atomic $\mathscr{L}_{\restriction\Sigma}$-formula, and for all $i \in [r]$, $\overline{u_i}(\overline{x})$ is a tuple of $\mathscr{L}_{\restriction\Pi}$-terms and f_i is a function symbol in $\mathcal{F}_\Sigma$ of the appropriate sort; and*

(ii) *for all $m \in \mathbb{N}$, all tuples $(f_1, \ldots, f_m)$ of functions $f_i \in \mathcal{F}_\Sigma$, and all tuples of $\mathscr{L}_{\restriction\Pi}$-terms $\overline{u_i}(\overline{x}, w)$, the formula*

$$\exists w \left[\bigwedge\!\!\!\!\bigwedge_{i=1}^{m} z_i = f_i(\overline{u_i}(\overline{x}, w)) \right]$$

is equivalent to a formula without Π-quantifiers, where z_i are appropriate Σ-variables.

Then T eliminates Π-quantifiers; equivalently (Lemma 2.1.7)*, T eliminates quantifiers relative to Σ. Moreover, if each of the formulas in* (ii) *is equivalent to an existential formula without Π-quantifiers, then every existential formula is equivalent modulo T to an existential formula without Π-quantifiers.*

Proof. By writing formulas in prenex normal form and inducting over Π-quantifiers, it suffices in turn by Lemma 2.1.8 and item (i) to show that

$$\exists w \left[\exists \overline{z} \left(\gamma(\overline{z}, \overline{y}) \wedge\!\!\!\wedge \bigwedge\!\!\!\!\bigwedge_{i=1}^{m} z_i = f_i(\overline{u_i}(\overline{x}, w)) \right) \right], \tag{2.3}$$

is equivalent to a formula without Π-quantifiers, where γ, f_i, and u_i are as in the formula (†) in Lemma 2.1.8. Since $\gamma(\overline{z}, \overline{y})$ is an $\mathscr{L}_{\restriction\Sigma}$-formula and w is a Π-variable, (2.3) is equivalent to

$$\exists \overline{z} \left[\gamma(\overline{z}, \overline{y}) \curlywedge \exists w \left(\bigcurlywedge_{i=1}^{m} z_i = f_i(\overline{u_i}(\overline{x}, w)) \right) \right], \tag{2.4}$$

and (2.4) is in turn equivalent to a formula without Π-quantifiers by item (ii), concluding thus the proof. The moreover part in the statement of the lemma is clear by construction. □

2.1.2 Recursive model theory

This subsection deals with the effective content of model theory needed for the correct formalization of the decidability statements that appear in the rest of the thesis, in particular those in Section 3.5; standard references for recursive model theory (also known as *computable* or *constructive* model theory) are [Mil99] and [Har98].

Albeit here it is assumed that the reader has a basic working knowledge of the concept of a (*partial*) *recursive function* (see for example [Her69], [Rog67], [Man10, Chapter V], [Mur99, Chapter 1]), the proofs which involve showing that a particular function or set is recursive[4] will always implicitly invoke the *Church-Turing thesis* (cf. [Man10, Chapter V, Subsections 2.5 and 2.6]) as it is commonly done in much of mathematical practice; that is, a function is recursive if and only if it can be computed somehow.

Definition 2.1.10 (cf. 2.1.1 in [Mil99])**.** Let $\mathscr{L}$ be a one-sorted language. Write $\mathscr{F}$ for the set of function symbols of $\mathscr{L}$, $\mathscr{R}$ for the set of relation symbols of $\mathscr{L}$, $\mathscr{C}$ for the set of constant symbols of $\mathscr{L}$, and $\mathscr{S}$ for the set of logical symbols of $\mathscr{L}$. The language $\mathscr{L}$ is *recursive* if there exists an injective function (called an *effective/recursive presentation of* $\mathscr{L}$)

$$\ulcorner - \urcorner : \mathscr{F} \,\dot\cup\, \mathscr{R} \,\dot\cup\, \mathscr{C} \,\dot\cup\, \mathscr{S} \longrightarrow \omega$$

such that

(i) if $\mathscr{V} := \{x_i \mid i < \omega\} \subseteq \mathscr{S}$ is the set of variables of $\mathscr{L}$, then $\ulcorner x_i \urcorner = 2i \in \omega$;

[4] Recall that a set $S \subseteq \omega^n$ is *recursive* if its characteristic function $\chi_S : \omega^n \longrightarrow \omega$ is (total) recursive.

(ii) the sets $\lceil \mathscr{F} \rceil$, $\lceil \mathscr{R} \rceil$, and $\lceil \mathscr{C} \rceil$ are recursive; and

(iii) the functions $\lceil \mathscr{F} \rceil \longrightarrow \omega$ and $\lceil \mathscr{R} \rceil \longrightarrow \omega$ sending $\lceil F \rceil$ to the arity of $F \in \mathscr{F}$ and $\lceil R \rceil$ to the arity of $R \in \mathscr{R}$ (respectively) are partial recursive and have recursive image.

Remark 2.1.11. Since every finite set is recursive, every finite one-sorted language is recursive. However, not every countable one-sorted language is recursive. Indeed, let $S \subseteq \omega$ be a non-recursive subset and consider the one-sorted language $\mathscr{L}$ consisting only of a single n-ary function symbol for each $n \in S$; then $\mathscr{L}$ is not recursive, since for any effective presentation $\lceil - \rceil$ of $\mathscr{L}$, the image of the arity map $\lceil \mathscr{F} \rceil \longrightarrow \omega$ is exactly S, hence not recursive.

Remark 2.1.12. Let $\mathscr{L}$ be a one-sorted recursive language and C be a new countable set of constant symbols; then the language $\mathscr{L}(C)$ obtained by enriching $\mathscr{L}$ with constant symbols for each $c \in C$ is also recursive. Indeed, let $\lceil - \rceil$ be an effective presentation of $\mathscr{L}$, and define $S \subseteq \omega$ to be the image of $\lceil - \rceil$, noting that S is recursive, and thus so is $\omega \setminus S$. If $\omega \setminus S$ is infinite, then one may extend the effective presentation of $\mathscr{L}$ to an effective presentation of $\mathscr{L}(C)$ by choosing any bijection $C \longrightarrow \omega \setminus S$. If $\omega \setminus S$ is not infinite, then define $\lceil - \rceil' := (2\,\cdot) \circ \lceil - \rceil$ where $2\,\cdot : \omega \hookrightarrow \omega$ is the function $n \mapsto 2n$; then $\lceil - \rceil'$ is an effective presentation of $\mathscr{L}$ whose recursive image in ω has infinite complement, therefore one can extend $\lceil - \rceil'$ to an effective presentation of $\mathscr{L}(C)$ as above.

Unless stated otherwise, $\mathscr{L}$ denotes a recursive one-sorted language for the remaining of this subsection.

Theorem 2.1.13. *Let* $\mathscr{L}$-Trm *and* $\mathscr{L}$-Fml *be the sets of $\mathscr{L}$-terms and $\mathscr{L}$-formulas, respectively. There exists an injective function (called a* Gödel numbering/coding of *$\mathscr{L}$)*

$$\ulcorner - \urcorner : \mathscr{L}\text{-Trm} \,\dot{\cup}\, \mathscr{L}\text{-Fml} \hookrightarrow \omega$$

such that $\ulcorner \mathscr{L}\text{-Trm} \urcorner$ *and* $\ulcorner \mathscr{L}\text{-Fml} \urcorner$ *are recursive.*

Proof. Folklore. The proof starts with an effective presentation of $\mathscr{L}$ and builds the Gödel numbering of $\mathscr{L}$ by induction on the complexity of terms and formulas, see for instance [Man10, Chapter VII, Section 4] or [EP84, Chapter 7, Section 38]. □

Definition 2.1.10 and Theorem 2.1.13 are commonly referred to in the literature as the *arithmetization* (or *Gödelization*) *of syntax*. The Gödel numbering of a recursive language $\mathscr{L}$ depends on the effective presentation of $\mathscr{L}$, but the choice of the effective presentation of $\mathscr{L}$ is not relevant to the purpose of tackling decidability issues dealt with in this thesis (see Remarks 2.1.16 and 2.1.18), therefore in what follows:

Unless stated otherwise, the symbol $\ulcorner - \urcorner$ stands for some fixed Gödel numbering of $\mathscr{L}$.

Using Theorem 2.1.13 one can also show that many other syntactic properties and constructions of $\mathscr{L}$-terms and $\mathscr{L}$-formulas are recursive; for instance, if $\mathscr{L}$-Sen is the set of $\mathscr{L}$-sentences, then $\ulcorner \mathscr{L}\text{-Sen} \urcorner$ is recursive and the function $\ulcorner \mathscr{L}\text{-Sen} \urcorner \times \ulcorner \mathscr{L}\text{-Sen} \urcorner \longrightarrow \ulcorner \mathscr{L}\text{-Sen} \urcorner$ given by $(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \mapsto \ulcorner \varphi \to \psi \urcorner$ is partial recursive with recursive image. Moreover, the Gödel numbering of $\mathscr{L}$ serves as a way of formalizing the effectiveness of syntactic notions, in particular:

Definition 2.1.14. Let $\Phi, \Psi \subseteq \mathscr{L}$-Fml and suppose that $\ulcorner \Phi \urcorner, \ulcorner \Psi \urcorner \subseteq \omega$ are recursive. Let T be an $\mathscr{L}$-theory. Say that every formula in Φ is *effectively equivalent* to a formula in Ψ modulo T if there exists a partial recursive function $F : \ulcorner \Phi \urcorner \longrightarrow \ulcorner \Psi \urcorner$ such that for every $\varphi(\overline{x}) \in \Phi$, the formula $\varphi(\overline{x})$ is equivalent to $\psi(\overline{x})$ modulo T, where $\psi(\overline{x})$ is the unique formula in Ψ such that $\ulcorner \psi(\overline{x}) \urcorner = F(\ulcorner \varphi(\overline{x}) \urcorner)$.

For example, if $\Phi := \mathscr{L}$-Fml and Ψ is the set of quantifier-free formulas, then Definition 2.1.14 says that T has *effective quantifier elimination*; note that there exist theories in finite (hence recursive) languages with quantifier elimination but without effective quantifier elimination, see [Pru01, Theorem 3].

Definition 2.1.15. A subset $\Sigma \subseteq \mathscr{L}$-Fml is *recursive* if $\ulcorner \Sigma \urcorner$ is recursive. An $\mathscr{L}$-theory T is *decidable* if T is recursive.

Intuitively, an $\mathscr{L}$-theory T is decidable if there exists an effective procedure which determines whether $\varphi \in T$ or not for every $\mathscr{L}$-sentence φ. More precisely, Definition 2.1.15 states that an $\mathscr{L}$-theory is decidable if there exists a recursive presentation $\lceil - \rceil$ of $\mathscr{L}$ such that the image of T under the Gödel numbering of $\mathscr{L}$ corresponding to $\lceil - \rceil$ is recursive. The next remark shows that for finite languages $\mathscr{L}$, decidability of an $\mathscr{L}$-theory T is independent of the choice of a recursive presentation of $\mathscr{L}$:

Remark 2.1.16. Suppose that $\mathscr{L}$ is finite, and that $\lceil - \rceil_1$ and $\lceil - \rceil_2$ are two recursive presentations of $\mathscr{L}$ yielding corresponding Gödel numberings $\ulcorner - \urcorner_1$ and $\ulcorner - \urcorner_2$ of $\mathscr{L}$. If T is an $\mathscr{L}$-theory and $\ulcorner T \urcorner_1$ is recursive, then $\ulcorner T \urcorner_2$ is also recursive. Indeed, since $\mathscr{L}$ is finite, it is clear that the images S_1 and S_2 of $\lceil - \rceil_1$ and $\lceil - \rceil_2$ are recursively isomorphic, that is, that there exists a partial recursive bijection $S_1 \longrightarrow S_2$ with partial recursive inverse. By the inductive construction of the Gödel numbering, this yields a recursive isomorphism $\ulcorner T \urcorner_1 \longrightarrow \ulcorner T \urcorner_2$, from which the claim follows.

Definition 2.1.17. Let $\mathscr{M}$ be a countable $\mathscr{L}$-structure (in particular $\mathscr{L}(M)$ is a recursive language by Remark 2.1.12).

(i) $\mathscr{M}$ is *recursive* if $\mathrm{Diag}_{\mathrm{at}}(\mathscr{M})$ is recursive, where

$$\mathrm{Diag}_{\mathrm{at}}(\mathscr{M}) := \{\varphi(\overline{a}) \in \mathscr{L}(M)\text{-Sen} \mid \varphi(\overline{x}) \text{ is an } \mathscr{L}\text{-literal}, \overline{a} \in M^{|\overline{x}|}, \text{ and } \mathscr{M} \models \varphi(\overline{a})\}$$

is the atomic diagram of $\mathscr{M}$ (recall that an *$\mathscr{L}$-literal* is an atomic $\mathscr{L}$-formula or the negation of an atomic $\mathscr{L}$-formula).

(ii) $\mathscr{M}$ is *decidable* if $\mathrm{Diag}_{\mathrm{el}}(\mathscr{M})$ is recursive, where

$$\mathrm{Diag}_{\mathrm{el}}(\mathscr{M}) := \{\varphi(\overline{a}) \in \mathscr{L}(M)\text{-Sen} \mid \varphi(\overline{x}) \text{ is an } \mathscr{L}\text{-formula}, \overline{a} \in M^{|\overline{x}|}, \text{ and } \mathscr{M} \models \varphi(\overline{a})\}$$

is the elementary diagram of $\mathscr{M}$. In other words, $\mathscr{M}$ is decidable if and only if the $\mathscr{L}(M)$-theory of $(\mathscr{M}, M)$ is decidable.

Remark 2.1.18. Spelled out in full, Definition 2.1.17 (i) says that a $\mathscr{L}$-structure $\mathscr{M}$ is recursive (decidable) if there exists a Gödel numbering $\ulcorner - \urcorner$ of $\mathscr{L}(M)$ such that $\ulcorner \mathrm{Diag}_{\mathrm{at}}(\mathscr{M}) \urcorner$ ($\ulcorner \mathrm{Diag}_{\mathrm{el}}(\mathscr{M}) \urcorner$) is recursive. In particular, if $\mathscr{M}$ is recursive (decidable) and $\mathscr{N}$ is an $\mathscr{L}$-structure $\mathscr{L}$-isomorphic to $\mathscr{M}$, then $\mathscr{N}$ is also recursive (decidable).

Remark 2.1.19. If $\mathscr{M}$ is recursive, then $\ulcorner M \urcorner$ is regarded in a canonical way as an $\mathscr{L}$-structure. The $\mathscr{L}$-isomorphism $\mathscr{M} \longrightarrow \ulcorner M \urcorner$ is known in the literature as a *recursive* (or *computable*) *presentation of* $\mathscr{M}$, see for example [KS99, Definition 2.1] and [Mon21, Definition I.1]. In general, if $\mathscr{M} \longrightarrow \ulcorner M \urcorner'$ is another recursive presentation of $\mathscr{M}$, then the $\mathscr{L}$-structures $\ulcorner M \urcorner$ and $\ulcorner M \urcorner'$ may have different computability-theoretic properties; for instance, $\ulcorner\mathrm{Diag}_{\mathrm{el}}(\mathscr{M})\urcorner$ may be recursive while $\ulcorner\mathrm{Diag}_{\mathrm{el}}(\mathscr{M})\urcorner'$ is not, see [KS99, Proposition 6.1] or [Mon21, Example I.1.2]. For the purposes of this thesis these issues will not arise as it will suffice to show that a recursive presentation with some property exists (cf. Remarks 2.1.16 and 2.1.18).

Remark 2.1.20. Let $\mathscr{M}$ be a countable $\mathscr{L}$-structure.

(i) $\mathscr{M}$ is recursive if and only if, after identifying M with $\ulcorner M\urcorner \subseteq \omega$, the interpretation of all function and relation symbols of $\mathscr{L}$ in $\ulcorner M\urcorner$ are (uniformly[5]) recursive.

(ii) Since the set of $\mathscr{L}$-literals is recursive, if $\mathscr{M}$ is decidable, then $\mathscr{M}$ is recursive; similarly, if $\mathscr{M}$ is decidable, then the $\mathscr{L}$-theory of $\mathscr{M}$ is also decidable. However, the converse of these two implications is not true in general: $(\omega, +, \cdot)$ is a recursive structure which is not decidable, and the structure $(\mathbb{R}, +, \cdot, -, 0, 1)$ is not recursive but it has a decidable theory.

(iii) Recursive and decidable structures in the sense of Definition 2.1.17 are also known in the literature as *constructive* and *strongly constructive* structures (respectively), see for example [EG98].

Example 2.1.21. Let $\mathscr{L}^{\mathrm{poring}} := \{+, -, \cdot, 0, 1, \leq\}$ be the language of partially ordered rings. A partially ordered ring A is *recursive* if it is a recursive $\mathscr{L}^{\mathrm{poring}}$-structure; for example, the totally ordered field $\mathbb{Q}$ is recursive. The Chapter 2 in Levin's thesis [Lev09] summarizes various constructions to obtain recursive totally ordered fields from a given recursive totally ordered domain; the main such constructions are:

(i) If A is a recursive totally ordered domain, then its (totally ordered) quotient field $\mathrm{qf}(A)$ is recursive; see [Lev09, Lemma 2.3.1].

(ii) If F is a recursive totally ordered field, then the polynomial ring $F[t]$ totally ordered by setting $0 < t < f$ for all $f \in F^{>0}$ (that is, t is a positive infinitesimal with respect to F) is recursive; see [Lev09, Example 2.3.2].

(iii) (Madison's theorem; [Mad70]) If F is a recursive totally ordered field, then its real closure $\rho(F)$ is recursive.

[5] For the exact definition of uniform recursivity of the interpretations of all function and relation symbols of a structure see [Ers+98, p. xv] or [Mon21, Definition 1.1]. Uniformity is only needed to define recursiveness of an $\mathscr{L}$-structure $\mathscr{M}$ whenever $\mathscr{L}$ is an infinite language.

In particular, the real closed field $\mathbb{R}_{\mathrm{alg}}$ of real algebraic numbers is recursive, and thus so is $\rho(\mathrm{qf}(\mathbb{R}_{\mathrm{alg}}[t])) = \rho(\mathbb{R}_{\mathrm{alg}}(t))$, where t is a positive infinitesimal with respect to $\mathbb{R}_{\mathrm{alg}}$.

Lemma 2.1.22. *Let A be a recursive ring, that is, A is a ring and a recursive $\mathscr{L}^{\mathrm{ring}}$-structure, where $\mathscr{L}^{\mathrm{ring}} := \{+, -, \cdot, 0, 1\}$. If $I \subseteq A$ is a recursive ideal (that is, I is an ideal of A and a recursive subset of A), then A/I is a recursive ring.*

Proof. Folklore, see for instance [Rab60, pp. 350, 1.6] or [ST99, Lemma 2.3.11]. □

The next result gives sufficient conditions for a recursive structure to be decidable; in particular, it follows by Proposition 2.1.23 that every recursive real closed field is decidable (this also follows by quantifier elimination for the $\mathscr{L}^{\mathrm{poring}}$-theory of real closed fields).

Proposition 2.1.23. *Let T be a decidable and model complete theory. If a model $\mathscr{M}$ of T is recursive, then $\mathscr{M}$ is decidable.*

Proof. Folklore; see [CMS21, Proposition 1] or [EG98, Proposition 1.5]. □

This section concludes with a discussion on the effective content of the material in Subsection 2.1.1. Suppose that $\mathscr{L}$ is a multi-sorted language, and assume for simplicity that $\mathscr{L}$ has finitely many sorts; this is only a mild assumption, since every multi-sorted language considered in the remaining part of the thesis has finitely many sorts. Under suitable modifications of Definition 2.1.10 it can also be defined what it means for the multi-sorted language $\mathscr{L}$ to be recursive; namely, one replaces items (i) and (ii) in Definition 2.1.10 by

(i)’ if $\mathscr{V}_S \subseteq \mathscr{S}$ is the set of S-sorted variables of $\mathscr{L}$, then $\ulcorner \mathscr{V}_S \urcorner$ is recursive for every sort S of $\mathscr{L}$, and

(ii)’ the sets $\ulcorner \mathscr{F} \urcorner$, $\ulcorner \mathscr{R} \urcorner$, and $\ulcorner \mathscr{C}_S \urcorner$ are recursive for every sort S of $\mathscr{L}$, where $\ulcorner \mathscr{C}_S \urcorner$ is the set of S-sorted constants.

Under this definition of recursiveness for the finitely-sorted language $\mathscr{L}$, the same proof of Theorem 2.1.13 can be adapted to prove that $\mathscr{L}$ also has a Gödel numbering, hence the notion of a recursive set of $\mathscr{L}$-sentences is well-defined. In particular, this enables the definition of an effective version of Definition 2.1.1 (v):

Definition 2.1.24. Let $\mathscr{L}$ be a finitely-sorted recursive language with a partition of sorts $\Pi \mathbin{\dot\cup} \Sigma$. An $\mathscr{L}$-theory T has *effective elimination of* Π*-quantifiers* if every $\mathscr{L}$-formula is effectively equivalent modulo T to a formula without Π-quantifiers.

Remark 2.1.25. The proofs of Lemma 2.1.6, Lemma 2.1.8, and Proposition 2.1.9 also show that the analogous effective versions of these statements also hold true whenever the language under consideration is finitely sorted and recursive.

2.2 Spectral spaces

Spectral spaces are a particular class of topological spaces. As mentioned at the beginning of this chapter, the relevance of these spaces in this thesis is that they can be used as spaces on which both real closed rings and lattice-ordered abelian groups admit functional representations, see Lemmas 2.3.3 and 2.4.7, respectively. An important result in this area is that the homeomorphism type of a spectral space X is completely determined by the isomorphism type of a bounded distributive lattice which is functorially associated to X. This is the content of *Stone duality for spectral spaces*, which is tangential to the use of spectral spaces which is made in Sections 2.3 and 2.4 below.

All the material present in this section can be found in the monograph [DST19]. Other references for the subject are [Joh82], as well as the more recent [GG24].

Definition 2.2.1. A topological space X is a *spectral space* if it satisfies the following four conditions:

(i) X is quasi-compact[6] and T_0.

(ii) The set

$$\mathring{\mathcal{K}}(X) := \{O \subseteq X \mid O \text{ is quasi-compact and open}\} \subseteq \mathcal{P}(X)$$

is a basis of open sets of X which is closed under finite intersections. In particular, $\mathring{\mathcal{K}}(X)$ is a bounded distributive lattice.

[6] Here a topological space is *quasi-compact* if every open cover has a finite subcover.

(iii) X is *sober*, that is, for every non-empty closed and irreducible subset C of X there is a point $x \in X$ with $C = \mathrm{cl}_X(\{x\})$, where $\mathrm{cl}_X(Y)$ is the closure of a set $Y \subseteq X$ in X.

Every *Boolean space* (also known as *Stone space*) is spectral. In fact:

Theorem 2.2.2. *Let X be a topological space. The following are equivalent:*

(i) *X is a Boolean space.*

(ii) *X is a spectral space and $\mathring{\mathcal{K}}(X)$ is a Boolean algebra.*

(iii) *X is a spectral space and Hausdorff.*

Proof. See [DST19, Theorem 1.3.4]. □

Every spectral space X has the bounded and distributive lattice $\mathring{\mathcal{K}}(X)$ as basis of open sets. Moreover, to every spectral space one can associate two other spectral topologies also having certain bounded and distributive lattices as basis of open sets:

Proposition 2.2.3. *Let X be a spectral space.*

(i) *The set*

$$\overline{\mathcal{K}}(X) := \{X \setminus U \mid U \in \mathring{\mathcal{K}}(X)\} \subseteq \mathcal{P}(X)$$

is a basis of open sets of a spectral topology on X, called the inverse topology*. The set X equipped with the inverse topology is denoted by X_{inv}, and the elements in $\overline{\mathcal{K}}(X)$ are called* closed constructible subsets of X*; moreover, $\mathring{\mathcal{K}}(X_{\mathrm{inv}}) = \overline{\mathcal{K}}(X)$ and $\overline{\mathcal{K}}(X_{\mathrm{inv}}) = \mathring{\mathcal{K}}(X)$.*

(ii) *The set*

$$\mathcal{K}(X) := \left\{ \bigcap_{i=1}^{n} (U_i \cup V_i) \mid n \in \mathbb{N},\ U_i \in \mathring{\mathcal{K}}(X) \text{ and } V_i \in \overline{\mathcal{K}}(X) \text{ for all } i \in [n] \right\}$$

is a basis of open sets of a spectral topology on X, called the constructible topology*. The set X equipped with the constructible topology is denoted by X_{con}, and the elements in $\overline{\mathcal{K}}(X)$ are called* constructible subsets of X*; moreover, $\mathring{\mathcal{K}}(X_{\mathrm{con}}) = \overline{\mathcal{K}}(X_{\mathrm{con}}) = \mathcal{K}(X)$ and X_{con} is a Boolean space.*

Proof. For item (i) see [DST19, Definition 1.4.1 and Theorem 1.4.3]. For item (ii) see [DST19, Definition 1.3.1, Theorem 1.3.14, and Corollary 1.3.15]. □

Definition 2.2.4. A function $f : X \longrightarrow Y$ between spectral spaces is a *spectral map* if $f^{-1}(U) \in \mathring{\mathcal{K}}(X)$ for every $U \in \mathring{\mathcal{K}}(Y)$. Spectral spaces together with spectral maps form a category which is denoted by **Spec**.

Remark 2.2.5. If X is a spectral space, then $\mathring{\mathcal{K}}(X)$ and $\overline{\mathcal{K}}(X)$ are bounded distributive lattices, and $\mathcal{K}(X)$ is a Boolean algebra. In particular, let **BDLat** be the category of bounded distributive lattices together with lattice homomorphisms which preserve the top and bottom elements, and let **BoolAlg** be the (full) subcategory of **BDLat** whose objects are Boolean algebras. Then (by taking preimages) $\mathring{\mathcal{K}}$ and $\overline{\mathcal{K}}$ are contravariant functors **Spec** $\longrightarrow$ **BDLat**, and $\mathcal{K}$ is a contravariant functor **Spec** $\longrightarrow$ **BoolAlg**.

Proposition 2.2.6. *Let X be a spectral space and $Y \subseteq X$ be any subset. The following are equivalent:*

(i) *Y is a* spectral subspace *of X, that is, the inclusion map $Y \overset{\subseteq}{\hookrightarrow} X$ is a spectral map.*

(ii) *Y is* proconstructible *in X, that is, Y is closed in X_{con}.*

Proof. See [DST19, Theorem 2.1.3]. □

2.2.1 The prime spectrum and the real spectrum of a ring

It is well known that the set $\mathrm{Spec}(A)$ of prime ideals of a ring A can be topologized with a spectral topology know as the *Zariski* (or *hull-kernel*) *topology*; in fact, by Hochster's theorem every spectral space is homeomorphic to $\mathrm{Spec}(A)$ for some ring A, see [Hoc69] or [DST19, Section 12.6]. The next statements summarize the main properties of the Zariski spectrum of a ring which will be needed in Section 2.3:

Notation 2.2.7. Let A be a ring. Write $\mathrm{Spec}(A)$ for the *prime* (or *Zariski*) *spectrum of* A, that is, the set of prime ideals of A. For each $a \in A$ set also $D(a) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid a \notin \mathfrak{p}\}$, $V(a) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid a \in \mathfrak{p}\}$, and $V(S) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid S \subseteq \mathfrak{p}\}$.

Proposition 2.2.8. *Let A be a ring.*

(i) *The sets $D(a)$ $(a \in A)$ form a basis of open sets for a spectral topology on* $\mathrm{Spec}(A)$ *called the* Zariski topology*; equipped with this topology, $\mathring{\mathcal{K}}(\mathrm{Spec}(A)) := \{\bigcup_{i=1}^{n} D(a_i) \mid n \in \mathbb{N},\ a_1, \dots, a_n \in A\}$ and $\overline{\mathcal{K}}(\mathrm{Spec}(A)) := \{V(S) \mid S \subseteq A \text{ finite}\}$.*

(ii) *If B is a ring and $f : A \longrightarrow B$ is a ring homomorphism, then* $\mathrm{Spec}(f) : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ *defined by* $\mathrm{Spec}(f)(\mathfrak{p}) := f^{-1}(\mathfrak{p})$ *for* $\mathfrak{p} \in \mathrm{Spec}(B)$ *is a spectral map; in particular,* $\mathrm{Spec}(-) : \mathbf{CRing} \longrightarrow \mathbf{Spec}$ *is a functor.*

Proof. See [DST19, Section 2.5.A] and [DST19, Section 12.1]. □

Proposition 2.2.9. *Let A be a ring. The map*

$$\begin{aligned} \overline{\mathcal{K}}(\mathrm{Spec}(A)) &\longrightarrow \mathcal{I}^{\mathrm{rad}}_{\mathrm{fin}}(A) \\ V(F) &\longmapsto \sqrt{(F)} \end{aligned}$$

is an anti-isomorphism[7] *of bounded and distributive lattices, where:*

(i) $\sqrt{(F)}$ *is the smallest radical ideal containing the ideal (F) generated by F; equivalently,*

$$\sqrt{(F)} = \{a \in A \mid \exists n \in \mathbb{N} \textit{ such that } a^n \in (F)\} = \bigcap_{\mathfrak{p} \in V(F)} \mathfrak{p} \qquad (*)$$

(ii) $(\mathcal{I}^{\mathrm{rad}}_{\mathrm{fin}}(A), \subseteq)$ *is the lattice of* finitely generated radical ideals *with join and meet of* $\sqrt{(F_1)}, \sqrt{(F_2)} \in \mathcal{I}^{\mathrm{rad}}_{\mathrm{fin}}(A)$ *given by* $\sqrt{(F_1) + (F_2)}$ *and* $\sqrt{(F_1)} \cap \sqrt{(F_2)}$ $(= \sqrt{(F_1) \cdot (F_2)})$ *(respectively), and top and bottom element given by* $\sqrt{(1)}$ *and* $\sqrt{(0)}$ *(respectively).*

Proof. See [DST19, Corollary 12.1.11]. The equalities in $(*)$ follow by basic commutative algebra, as well the fact that if $F_1, F_2 \subseteq A$ are finite, then the ideals $(F_1) + (F_2)$ and $(F_1) \cdot (F_2)$ are finitely generated. □

In [CR82] Coste and Roy introduce another spectral space functorially associated to any ring A, namely the *real spectrum* $\mathrm{Sper}(A)$ *of* A, see Definition 2.2.10 (II) and Proposition 2.2.13. The elements in $\mathrm{Sper}(A)$ are *prime cones of* A, which are certain subsets of A that are in bijective correspondence with pairs $(\mathfrak{p}, \leq)$, where $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\leq$ is a total order on $A/\mathfrak{p}$ turning it into a totally ordered domain, see Definition 2.2.10 (I) and Proposition 2.2.12. It is in this way that $\mathrm{Sper}(A)$ captures order-theoretic information of A.

Definition 2.2.10. Let A be a ring.

[7] An *anti-isomorphism of posets* $F : (P, \leq_P) \longrightarrow (Q, \leq_Q)$ is a bijection F between posets $(P, \leq_P)$ and $(Q, \leq_Q)$ such that $f \leq_P g$ if and only if $F(f) \geq_Q F(g)$ for all $f, g \in P$.

(I) A *prime cone of* A is a subset $\alpha \subseteq A$ satisfying the following properties:

(i) If $a, b \in \alpha$, then $a + b \in \alpha$ and $ab \in \alpha$.

(ii) If $a \in A$, then $a^2 \in \alpha$.

(iii) For all $a \in A$, either $a \in \alpha$ or $-a \in \alpha$.

(iv) $-1 \notin \alpha$.

(v) The set $\operatorname{supp}(\alpha) := \alpha \cap -\alpha = \{a \in A \mid a \in \alpha \text{ and } -a \in \alpha\}$, called the *support of* α, is a prime ideal of A.

(II) The set

$$\operatorname{Sper}(A) := \{\alpha \subseteq A \mid \alpha \text{ is a prime cone of } A\}$$

is the *real spectrum of* A.

Remark 2.2.11. Let A be a ring and $\alpha \in \operatorname{Sper}(A)$. Then $\operatorname{supp}(\alpha)$ is a *real ideal of* A, that is, for all $n \in \mathbb{N}$ and all $a_1, \dots, a_n \in A$, $\sum_{i=1}^n a_i^2 \in \operatorname{supp}(\alpha)$ implies $a_i \in \operatorname{supp}(\alpha)$ for all $i \in [n]$. Indeed, pick $i \in [n]$. Since $\operatorname{supp}(\alpha)$ is a prime ideal, it suffices to show that $a_i^2 \in \operatorname{supp}(\alpha) = \alpha \cap -\alpha$, and since $a_i^2 \in \alpha$ by Definition 2.2.10 (I) (ii), it suffices to show in turn that $a_i^2 \in -\alpha$, that is, $-a_i^2 \in \alpha$. Since $\sum_{i=1}^n a_i^2 \in \operatorname{supp}(\alpha) \subseteq -\alpha$, one has $-a_1^2 - \dots - a_n^2 \in \alpha$, therefore

$$-a_i^2 = -a_1^2 - \dots - a_n^2 + \sum_{i \in [n] \setminus \{i\}} a_i^2 \in \alpha$$

by items (II) (i) and (II) (iii) in Definition 2.2.10, as required.

Proposition 2.2.12. *Let* A *be a ring.*

(i) *If* $\mathfrak{p}$ *is a prime ideal and* $\leq$ *is a total order on* $A/\mathfrak{p}$ *such that* $(A/\mathfrak{p}, \leq)$ *is a totally ordered ring, then the set* $\{a \in A \mid a/\mathfrak{p} \geq 0/\mathfrak{p}\}$ *is a prime cone of* A.

(ii) *The set*

$$\{(\mathfrak{p}, \leq) \mid \mathfrak{p} \in \operatorname{Spec}(A) \text{ and } (A/\mathfrak{p}, \leq) \text{ is a totally ordered ring}\}$$

is in bijection with $\operatorname{Sper}(A)$*: the bijection sends* $(\mathfrak{p}, \leq)$ *to* $\{a \in A \mid a/\mathfrak{p} \geq 0/\mathfrak{p}\}$*, and its inverse sends* $\alpha \in \operatorname{Sper}(A)$ *to* $(\operatorname{supp}(\alpha), \leq_\alpha)$*, where*

$$a/\operatorname{supp}(\alpha) \leq_\alpha b/\operatorname{supp}(\alpha) \overset{\text{def}}{\iff} b - a \in \alpha.$$

Proof. See [DST19, Proposition 13.1.4]. □

Proposition 2.2.13. *Let A be a ring.*

(i) *The sets*

$$H^{>0}(a_1, \dots, a_n) := \{\alpha \in \mathrm{Sper}(A) \mid -a_i \notin \alpha \text{ for all } i \in [n]\}$$

for all finite sequences $a_1, \dots, a_n \in A$ ($n \in \mathbb{N}$) form a basis of open sets for a spectral topology on Sper(A) *called the* Harrison topology*; equipped with this topology, the quasi-compact open subsets of* Sper(A) *are exactly the finite unions of sets of the form above.*

(ii) *If B is a ring and $f : A \longrightarrow B$ is a ring homomorphism, then* $\mathrm{Sper}(f) : \mathrm{Sper}(B) \longrightarrow \mathrm{Sper}(A)$ *defined by* $\mathrm{Sper}(f)(\alpha) := f^{-1}(\alpha)$ *for $\alpha \in \mathrm{Sper}(B)$ is a spectral map; in particular,* $\mathrm{Sper}(-) : \mathbf{CRing} \longrightarrow \mathbf{Spec}$ *is a functor.*

Proof. See [DST19, Theorem 2.5.8]. □

The connection between the Zariski and the real spectrum of a ring is captured in the following:

Proposition 2.2.14. *Let A be a ring. The map* $\mathrm{supp}(-) : \mathrm{Sper}(A) \longrightarrow \mathrm{Spec}(A)$ *given by $\alpha \mapsto \mathrm{supp}(\alpha)$ is spectral; moreover,* supp *is a natural transformation of functors* $\mathrm{Sper} \longrightarrow \mathrm{Spec}$.

Proof. See [DST19, Theorem 2.5.12. (ii)]. □

2.3 Real closed rings

The original construction of real closed rings given by Schwartz in [Sch89] assigns a ring $\rho(A, X)$ to every ring A and to every proconstructible subset X of the real spectrum $\mathrm{Sper}(A)$ of A. The ring $\rho(A, X)$ is the *real closure of A on X*, and it is a particular subring of the product of real closed fields $\prod_{\alpha \in X} \mathrm{qf}(A/\mathrm{supp}(\alpha))$, namely, $\rho(A, X)$ is the ring of *compatible and constructible sections on X*, see [Sch89, Definition I.2.8].

An arbitrary ring A is then defined to be *real closed* if the canonical map $a \mapsto (a/\mathrm{supp}(\alpha))_{\alpha \in \mathrm{Sper}(A)}$ yields a ring isomorphism $A \longrightarrow \rho(A, \mathrm{Sper}(A))$ (see [Sch89, Definition I.4.1.]), therefore a ring is defined to be real closed if and only if it is canonically isomorphic to the ring of constructible and compatible sections on its real spectrum. The following is an equivalent definition of real closed rings which can be found in [Sch86] and [PS].

Definition 2.3.1. A ring A is a *real closed ring* if it satisfies the following conditions:

(i) A is reduced, that is, if $a^2 = 0$ implies $a = 0$ for all $a \in A$ (equivalently, the intersection of all prime ideals is the zero ideal);

(ii) the set of squares of A is the set of non-negative elements of a partial order $\leq$ on A and $(A, \leq)$ is an *f-ring*, i.e., $(A, \leq)$ is a partially ordered ring such that for every $a, b \in A$ the supremum $a \vee b$ and the infimum $a \wedge b$ exist in A, and $a \wedge b = 0$ and $c \geq 0$ imply $(ca) \wedge b = 0$ for all $a, b, c \in A$;

(iii) for all $a, b \in A$, if $0 \leq a \leq b$, then there exists $c \in A$ such that $bc = a^2$; and

(iv) $\mathrm{qf}(A/\mathfrak{p})$ is a real closed field and $A/\mathfrak{p}$ is integrally closed for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

It is clear from Definition 2.3.1 that a field is a real closed ring if and only if it is a real closed field. Arbitrary convex subrings of real closed fields are also real closed rings: these are precisely those real closed rings which are valuation rings (see Theorem 2.3.6), and their main algebraic and model-theoretic properties are summarized in Subsection 2.3.1. Other examples of real closed rings include:

- For every ring A and every $X \subseteq \mathrm{Sper}(A)$, the real closure $\rho(A, X)$ of A on X, see [Sch89, Theorem I.3.25].
- Rings $C(X)$ of continuous real-valued functions on a topological space X, see [Sch97, Theorem 1.2].
- Rings $C_{\text{s.a.}}(X)$ of continuous semi-algebraic functions $X \longrightarrow R$ on a semi-algebraic subset $X \subseteq R^m$ over a real closed field R, see [Sch89, Section III.1].
- Semi-algebraic function rings in the sense of Madden and Schwartz, see [SM99, Section 7 and Example 12.15].

Many of the examples of real closed rings given above are rings of functions. Lemma 2.3.3 below shows how to regard every real closed ring as an actual ring of functions on its Zariski spectrum. Before proving that, the main properties of real closed rings that will be made use of in the rest of this document are summarized in the following theorem:

Theorem 2.3.2. (I) *The category of real closed rings together with ring homomorphisms is complete and cocomplete; in particular, direct and fibre products of real closed rings are real closed.*

(II) *Let A be a real closed ring.*

(i) *If $I \subseteq A$ is an ideal, then A/I is real closed if and only if I is radical.*

(ii) *If $S \subseteq A$ is a multiplicative subset, then the localization $S^{-1}A$ is real closed.*

(iii) *The poset $(\mathrm{Spec}(A), \subseteq)$ is a* root system, *i.e., for all $\mathfrak{p} \in \mathrm{Spec}(A)$, the principal up-set $\mathfrak{p}^{\uparrow} := \{\mathfrak{q} \in \mathrm{Spec}(A) \mid \mathfrak{p} \subseteq \mathfrak{q}\}$ is a chain.*

(iv) *If $I, J \subseteq A$ are radical ideals, then $I + J$ is a radical ideal. In particular:*

(a) *If $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$ and $1 \notin \mathfrak{p} + \mathfrak{q}$, then $\mathfrak{p} + \mathfrak{q} \in \mathrm{Spec}(A)$.*

(b) *The poset $(\mathcal{I}^{\mathrm{rad}}(A), \subseteq)$ of radical ideals of A is a distributive lattice with join and meet operations given by sum and intersection of ideals, respectively.*

(III) *If A and B are real closed rings, then any ring homomorphism $f : A \longrightarrow B$ preserves the order and the lattice operations $\vee$ and $\wedge$ (see* Definition 2.3.1 (ii)*).*

Proof. (I). By [SM99, Section 12], the category of real closed rings is a monoreflective subcategory of the category of reduced partially ordered rings; since the latter category is complete and cocomplete by [SM99, Theorem 1.7], so is the category of real closed rings by [SM99, Proposition 2.3] and [SM99, Proposition 2.7].

(II). Item (i) is [Sch89, Chapter I, Theorem 4.5] and item (ii) is [SM99, Proposition 12.6]. Item (iii) follows from [DST19, Theorem 13.1.9 (iii)] together with the fact that the support map $\mathrm{Sper}(A) \longrightarrow \mathrm{Spec}(A)$ is a homeomorphism (see [Sch89, Chapter I, Theorem 3.10] or [SM99, Proposition 12.4. (d)]), and (iv) is [Sch86, Corollary 15]. For (iv) (a), pick $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$ such that $1 \notin \mathfrak{p} + \mathfrak{q}$. Since $\mathfrak{p} + \mathfrak{q}$ is a radical ideal, $\mathfrak{p} + \mathfrak{q} = \bigcap\{\mathfrak{r} \in \mathrm{Spec}(A) \mid \mathfrak{p} + \mathfrak{q} \subseteq \mathfrak{r}\}$, and since $1 \notin \mathfrak{p} + \mathfrak{q}$, $\mathcal{S} := \{\mathfrak{r} \in \mathrm{Spec}(A) \mid \mathfrak{p} + \mathfrak{q} \subseteq \mathfrak{r}\} \neq \varnothing$. If $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathcal{S}$, then $\mathfrak{p} \subseteq \mathfrak{r}_1, \mathfrak{r}_2$, therefore $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are comparable under subset inclusion by (iii); it follows that $\mathcal{S}$ is a chain in $(\mathrm{Spec}(A), \subseteq)$, therefore $\mathfrak{p} + \mathfrak{q} = \bigcap_{\mathfrak{r} \in \mathcal{S}} \mathfrak{r} \in \mathrm{Spec}(A)$. For (iv) (b) it remains to show that the lattice $(\mathcal{I}^{\mathrm{rad}}(A), +, \cap)$ is distributive, so pick $I, J, K \in \mathcal{I}^{\mathrm{rad}}(A)$; the inclusion $(I \cap J) + (I \cap K) \subseteq I \cap (J + K)$ is clear, and if $a \in I \cap (J + K)$, then $a = j + k$ for some $j \in J$ and $k \in K$, therefore $a^2 = aj + ak \in IJ + IK \subseteq (I \cap J) + (I \cap K)$, and since $(I \cap J) + (I \cap K) \in \mathcal{I}^{\mathrm{rad}}(A)$, $a \in (I \cap J) + (I \cap K)$ follows, as required.

(III). That f preserves the order is clear by Definition 2.3.1 (ii), and that f preserves the lattice operations follows from [DM95, Lemma 2.2]. □

Lemma 2.3.3. *Let A be a real closed ring. Then A is isomorphic (as a ring and as a lattice) to an f-ring of functions $\mathrm{Spec}(A) \longrightarrow R$ for some real closed field R.*

Proof. Since A is reduced, $\bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p} = (0)$, therefore the canonical map $A \longrightarrow \prod_{\mathfrak{p} \in \mathrm{Spec}(A)} A/\mathfrak{p}$ given by $a \mapsto (a/\mathfrak{p})_{\mathfrak{p} \in \mathrm{Spec}(A)}$ is injective. The canonical map $A \longrightarrow \prod_{\mathfrak{p} \in \mathrm{Spec}(A)} A/\mathfrak{p}$ is a ring homomorphism, and it preserves the lattice operations by Theorem 2.3.2 (III) since $\prod_{\mathfrak{p} \in \mathrm{Spec}(A)} A/\mathfrak{p}$ is real closed by items (II) (i) and (I) in Theorem 2.3.2. Since the $\mathscr{L}^{\mathrm{poring}}$-theory of real closed fields is complete and has quantifier elimination, this theory has the joint embedding property by [CK90, Proposition 3.5.11], therefore[8] there exists a real closed field R such that $\mathrm{qf}(A/\mathfrak{p}) \subseteq R$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$, and thus the composite map

$$A \longrightarrow \prod_{\mathfrak{p} \in \mathrm{Spec}(A)} A/\mathfrak{p} \xrightarrow{\subseteq} \prod_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathrm{qf}(A/\mathfrak{p}) \xrightarrow{\subseteq} \prod_{\mathfrak{p} \in \mathrm{Spec}(A)} R = R^{\mathrm{Spec}(A)}$$

is a ring and a lattice isomorphism onto its image, as required. □

The next proposition can be seen as a special case of [DM95, Section 4]; there it is shown that the lattice of quasi-compact open sets of the *Brumfield spectrum* of an f-ring A is isomorphic to the lattice of *principal radical ℓ-ideals* of A.

Proposition 2.3.4. *Let A be a real closed ring. Then $\overline{\mathcal{K}}(\mathrm{Spec}(A)) = \{V(a) \mid a \in A\}$ and the map*

$$\begin{aligned} \overline{\mathcal{K}}(\mathrm{Spec}(A)) &\longrightarrow \mathcal{I}^{\mathrm{rad}}_{\mathrm{prin}}(A) \\ V(a) &\longmapsto \sqrt{(a)} \end{aligned}$$

[8] If a model complete theory T has the joint embedding property (such as the theory of real closed fields), then an arbitrary collection of models $\{M_i\}_{i \in I}$ of T can be jointly embedded into a model M of T: M is constructed by transfinite recursion, and it is a model of T by the Elementary Chain Theorem ([CK90, Theorem 3.1.9]).

is an anti-isomorphism of bounded distributive lattices, where $(\mathcal{I}^{\text{rad}}_{\text{prin}}(A), \subseteq)$ *is the lattice of* principal radical ideals *with join and meet of* $\sqrt{(a_1)}, \sqrt{(a_2)} \in \mathcal{I}^{\text{rad}}_{\text{prin}}(A)$ *given by* $\sqrt{(a_1)} + \sqrt{(a_2)}$ $(= \sqrt{(a_1^2 + a_2^2)})$ *and* $\sqrt{(a_1)} \cap \sqrt{(a_2)}$ $(= \sqrt{(a_1 \cdot a_2)})$ *(respectively), and top and bottom element given by* $\sqrt{(1)}$ *and* $\sqrt{(0)}$ *(respectively).*

Proof. Elements in $\overline{\mathcal{K}}(\text{Spec}(A))$ are of the form $V(F)$ (see Notation 2.2.7), where $F \subseteq A$ is finite, say $F := \{a_1, \dots, a_n\}$. Since A is real closed, it support map $\text{Sper}(A) \longrightarrow \text{Spec}(A)$ is a homeomorphism (see [Sch89, Chapter I, Theorem 3.10] or [SM99, Proposition 12.4. (d)]), therefore every $\mathfrak{p} \in \text{Spec}(A)$ is $\text{supp}(\alpha)$ for a unique $\alpha \in \text{Sper}(A)$. In particular, every $\mathfrak{p} \in \text{Spec}(A)$ is a real ideal by Remark 2.2.11, therefore (cf. [PS, p. 13])

$$\begin{aligned} V(F) = V(\{a_1, \dots, a_n\}) &= \{\mathfrak{p} \mid a_1, \dots, a_n \in \mathfrak{p}\} \\ &= \{\mathfrak{p} \mid a_1^2, \dots, a_n^2 \in \mathfrak{p}\} \\ &= \{\mathfrak{p} \mid a_1^2 + \dots + a_n^2 \in \mathfrak{p}\} \\ &= V(a_1^2 + \dots + a_n^2), \end{aligned} \tag{$*$}$$

from which $\overline{\mathcal{K}}(\text{Spec}(A)) = \{V(a) \mid a \in A\}$ follows. Note that $(*)$ also implies that $V(a) \cap V(b) = V(\{a, b\}) = V(a^2 + b^2)$ and that if $F = \{a_1, \dots, a_n\}$, then $\sqrt{(F)} = \sqrt{(a_1^2 + \dots + a_n^2)} \in \mathcal{I}^{\text{rad}}_{\text{prin}}(A)$ by Proposition 2.2.9 (i), therefore by Proposition 2.2.9 it suffices to show that the join operation on $\mathcal{I}^{\text{rad}}_{\text{fin}}(A)$ defined in Proposition 2.2.9 (ii) and the join operation on $\mathcal{I}^{\text{rad}}_{\text{prin}}(A)$ defined in the statement of this lemma coincide, that is, that

$$\sqrt{(a_1) + (a_2)} \overset{(1)}{=} \sqrt{(a_1^2 + a_2^2)} \overset{(2)}{=} \sqrt{(a_1)} + \sqrt{(a_2)}$$

for all $a_1, a_2 \in A$. The equality (1) follows from $V(a_1) \cap V(a_2) = V(a_1^2 + a_2^2)$ and from the fact that the map in Proposition 2.2.9 is an anti-isomorphism. Since $(a_1), (a_2) \subseteq (a_1) + (a_2)$, it follows that $\sqrt{(a_1)}, \sqrt{(a_2)} \subseteq \sqrt{(a_1) + (a_2)}$, therefore $\sqrt{(a_1)} + \sqrt{(a_2)} \subseteq \sqrt{(a_1) + (a_2)}$, but $\sqrt{(a_1)} + \sqrt{(a_2)}$ is a radical ideal by Theorem 2.3.2 (II) (iv) and it contains $(a_1) + (a_2)$, therefore $\sqrt{(a_1)} + \sqrt{(a_2)} = \sqrt{(a_1) + (a_2)}$ by minimality of $\sqrt{(a_1) + (a_2)}$. This concludes the proof. ∎

2.3.1 Real closed valuation rings

Recall that a ring A is a *valuation ring* if A is a domain, and for all non-zero $a, b \in \mathrm{qf}(A)$, either $a/b \in A$ or $b/a \in A$; a valuation ring A is *non-trivial* if $A \neq \mathrm{qf}(A)$. Valuation rings are in particular local rings whose unique maximal ideal consists of those elements which are not multiplicative units of the ring; all of this is contained in [EP05].

Definition 2.3.5. A ring A is a *real closed valuation ring* if A is a real closed ring and a valuation ring.

Theorem 2.3.6. *Let A be a ring. The following are equivalent:*

(I) *A is a real closed valuation ring.*

(II) $\mathrm{qf}(A)$ *is a real closed field and A is convex in* $\mathrm{qf}(A)$.

(III) $\mathrm{qf}(A)$ *is a real closed field and* $A = \{a \in \mathrm{qf}(A) \mid v(a) \geq 0\}$, *where* $v : \mathrm{qf}(A) \twoheadrightarrow \Gamma \cup \{\infty\}$ *is an* order-compatible valuation *on* $\mathrm{qf}(A)$, *that is, v is a valuation on* $\mathrm{qf}(A)$ (see [EP05]) *such that for all $a, b \in A$,* $0 \leq a \leq b$ *implies* $v(b) \leq v(a)$.

(IV) *A is a valuation ring, and both* $\mathrm{qf}(A)$ *and* $A/\mathfrak{m}_A$ *is are real closed fields.*

(V) *A is a totally ordered domain which satisfies the intermediate value property for polynomials in one variable.*

(VI) *A is a totally ordered domain which satisfies the following conditions:*

  (i) *For all $a, b \in A$, if $0 < a < b$, then there exists $c \in A$ such that $bc = a$.*

  (ii) *Every positive element has a square root.*

  (iii) *Every monic polynomial of odd degree has a root.*

Proof. (I) $\Rightarrow$ (II). It suffices to show that A is convex in $\mathrm{qf}(A)$, so pick $a \in A$ and $b \in \mathrm{qf}(A)$ such that $0 < b < a$ and assume for contradiction that $b \notin A$; then $b^{-1} \in A$, therefore $0 < 1 < ab^{-1}$ in A implies that there exists $c \in A$ such that $ab^{-1}c = 1^2 = 1$ (Definition 2.3.1 (iii)), hence $b = ac \in A$, a contradiction.

(II) $\Leftrightarrow$ (IV). By [KS22, Proposition 2.2.4] and [KS22, Theorem 2.5.1 (b)].

(IV) $\Rightarrow$ (I). Pick $\mathfrak{p} \in \mathrm{Spec}(A)$. A is convex in $\mathrm{qf}(A)$ by the implication (IV) $\Rightarrow$ (II), therefore $A_{\mathfrak{p}}$ is also convex in $\mathrm{qf}(A)$; in particular, $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is a real closed field by the implication (II) $\Rightarrow$ (IV), therefore $A/\mathfrak{p}$ is a valuation ring (hence integrally closed) of the real closed field $\mathrm{qf}(A/\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, and thus it remains to show that the condition in item (iii) of Definition 2.3.1 is satisfied. Pick $a, b \in A$ such that $0 \leq a \leq b$; then $0 \leq a^2/b \leq b^2/b = b \in A$, and since A is convex in $\mathrm{qf}(A)$, it follows that $c := a^2/b \in A$ is such that $bc = a^2$, as required.

(II) $\Leftrightarrow$ (III) $\Leftrightarrow$ (V) $\Leftrightarrow$ (VI) See [CD83, Theorem 1 and Lemma 4]. □

A particular class of real closed valuation rings are those valuation rings corresponding to the canonical valuation of real closed Hahn series fields:

Definition 2.3.7. Let $\boldsymbol{k}$ be a field and Γ be a totally ordered abelian group. Define $\boldsymbol{k}((\Gamma)) := \boldsymbol{k}((x^{\Gamma}))$ to be the set of formal series $a = \sum a_{\gamma}x^{\gamma} := \sum_{\gamma\in\Gamma} a_{\gamma}x^{\gamma}$ where $\mathrm{supp}(a) := \{\gamma \in \Gamma \mid a_{\gamma} \neq 0\}$ is a well-ordered subset of Γ.

Theorem 2.3.8. *Let $\boldsymbol{k}$ be a field and Γ be a totally ordered abelian group.*

(i) *The set $\boldsymbol{k}((\Gamma))$ endowed with the operations of pointwise addition and Cauchy product of formal series*

$$\sum a_{\gamma}x^{\gamma} + \sum b_{\gamma}x^{\gamma} := \sum (a_{\gamma} + b_{\gamma})x^{\gamma}$$

and

$$\left(\sum a_{\gamma}x^{\gamma}\right)\left(\sum b_{\gamma}x^{\gamma}\right) := \sum_{\gamma\in\Gamma}\sum_{\delta+\varepsilon=\gamma}(a_{\delta}b_{\varepsilon})x^{\gamma},$$

respectively, is a field called Hahn series field.

(ii) *The map $\nu : \boldsymbol{k}((\Gamma)) \twoheadrightarrow \Gamma \cup \{\infty\}$ given by $\nu(a) := \min(\mathrm{supp}(a))$ if $a \neq 0$ and $\nu(0) = \infty$ is a valuation with residue field $\boldsymbol{k}$; write $\boldsymbol{k}[[\Gamma]] := \boldsymbol{k}((\Gamma^{\geq 0}))$ for its corresponding valuation ring.*

(iii) *If $\boldsymbol{k}$ is a totally ordered field, then $\boldsymbol{k}((\Gamma))$ has the structure of a totally ordered field by setting $a > 0$ if and only if $a_{\nu(a)} > 0$ for all $a \in \boldsymbol{k}((\Gamma))$; under this total order, ν is an order-compatible valuation on $\boldsymbol{k}((\Gamma))$.*

(iv) *$\boldsymbol{k}((\Gamma))$ is a real closed field if and only if $\boldsymbol{k}$ is real closed and Γ is divisible; in particular, $\boldsymbol{k}[[\Gamma]]$ is a real closed valuation ring if and only if $\boldsymbol{k}$ is real closed and Γ is divisible.*

Proof. Folklore, see for instance [EP05, Exercise 3.5.6], [ADH17, Section 3.5], or [DW96, Section 2]; the last statement in (iv) follows from the equivalence (I) $\Leftrightarrow$ (III) in Theorem 2.3.6. $\square$

It is well known that if V is a non-trivial real closed valuation ring, then there exists a local embedding[9] $V \hookrightarrow \boldsymbol{k}[[\Gamma]]$, where $\boldsymbol{k} := V/\mathfrak{m}_V$ and $\Gamma := \mathrm{qf}(V)^\times/V^\times$ ([Pri83, 62, Satz 21] and Theorem A.4), and $\boldsymbol{k}[[\Gamma]]$ is a real closed valuation ring by the implication (I) $\Rightarrow$ (IV) in Theorem 2.3.6; in fact, one can do slightly better:

Proposition 2.3.9. *Let $V \subseteq W$ be a local embedding of non-trivial real closed valuation rings, and set $\boldsymbol{k} := V/\mathfrak{m}_V$, $\boldsymbol{l} := W/\mathfrak{m}_W$, $\Gamma := \mathrm{qf}(V)^\times/V^\times$, and $\Delta := \mathrm{qf}(W)^\times/W^\times$, noting that $V \subseteq W$ induces a canonical embedding $\boldsymbol{k}[[\Gamma]] \subseteq \boldsymbol{l}[[\Delta]]$. There exist local embeddings of non-trivial real closed valuation rings $\varepsilon_V : V \hookrightarrow \boldsymbol{k}[[\Gamma]]$ and $\varepsilon_W : W \hookrightarrow \boldsymbol{l}[[\Delta]]$ such that $\varepsilon_{W\restriction V} = \varepsilon_V$.*

Proof. Immediate from Theorem A.5, since $V \subseteq W$ being a local embedding implies that $(\mathrm{qf}(V), V) \subseteq (\mathrm{qf}(W), W)$ is an embedding of real closed valued fields. $\square$

Lemma 2.3.10. *Let V be a non-trivial real closed valuation ring, $\lambda : V \twoheadrightarrow V/\mathfrak{m}_V =: \boldsymbol{k}$ be the residue map, $B \subseteq \boldsymbol{k}$ be a subring. The ring $A := \lambda^{-1}(B) \subseteq V$ is a real closed valuation ring if and only of B is a real closed valuation ring.*

Proof. Straightforward from Theorem 2.3.2 (I) and the equivalence (I) $\Leftrightarrow$ (II) in Theorem 2.3.6, using also the fact that λ is an order-preserving map and that $\lambda^{-1}(B) \cong V \times_{\boldsymbol{k}} B$. $\square$

This subsection concludes with some model-theoretic facts about real closed valuation rings which will be needed for Chapters 3 and 4. For what follows, recall that $\mathscr{L}^{\mathrm{ring}} := \{+, -, \cdot, 0, 1\}$ is the language of rings.

Lemma 2.3.11. *If V and W are valuation rings such that $V \subseteq W$, then $V \subseteq W$ as $\mathscr{L}^{\mathrm{ring}}(\mathtt{m})$-structures if and only if $V \subseteq W$ as $\mathscr{L}^{\mathrm{ring}}(\mathrm{div})$-structures, where $\mathtt{m}$ is a unary predicate interpreted as the maximal ideal and* div *is a binary predicate interpreted as the divisibility relation.*

[9] Recall that an embedding $f : A \hookrightarrow B$ of local rings is said to be local if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$, where $\mathfrak{m}_A$ and $\mathfrak{m}_B$ are the unique maximal ideals of A and B, respectively.

Proof. If $V \subseteq W$ as $\mathscr{L}^{\text{ring}}(\text{div})$-structures and $a \in \mathfrak{m}_V$, then $V \models \neg\text{div}(a,1)$, hence $W \models \neg\text{div}(a,1)$ and thus $V \cap \mathfrak{m}_W = \mathfrak{m}_V$. Suppose now that $V \subseteq W$ as $\mathscr{L}^{\text{ring}}(\mathtt{m})$-structures and let $a, b \in V$ be such that $W \models \text{div}(a,b)$, so that there exists $c \in W$ such that $ac = b$, and assume for contradiction that $V \models \neg\text{div}(a,b)$; in particular, $b \neq 0$ and $c \notin V$. Since V is a valuation ring, $V \models \text{div}(b,a)$, therefore there exists $d \in V$ such that $bd = a$; in particular, $ac = bdc = b$, hence $b(1-dc) = 0$, therefore $1 = dc$ and thus $c^{-1} = d \in V$, so $c^{-1} \in \mathfrak{m}_V = \mathfrak{m}_W \cap V$, a contradiction to $c, c^{-1} \in W$. □

Remark 2.3.12. The proof of Lemma 2.3.11 also shows that if A and B are local rings such that $A \subseteq B$ as $\mathscr{L}^{\text{ring}}(\text{div})$-structures, then $A \subseteq B$ as $\mathscr{L}^{\text{ring}}(\mathtt{m})$-structures, i.e., $A \subseteq B$ is a local embedding.

In [CD83] the model theory of non-trivial real closed rings is studied in the languages $\mathscr{L}^{\text{ring}}(\leq)$ and $\mathscr{L}^{\text{ring}}(\leq, \text{div})$; in particular, since the class of non-trivial real closed valuation rings is elementary in the language $\mathscr{L}^{\text{ring}}(\leq)$ and the total order $x \leq y$ in structures of this class is defined by the existential formula $\exists z[z = (y-x)^2]$ (Theorem 2.3.6), the class of non-trivial real closed valuation rings is also elementary in the language of rings.

Proposition 2.3.13. *Let* RCVR *be the* $\mathscr{L}^{\text{ring}}$*-theory of non-trivial real closed valuation rings, and* $\mathtt{p}$ *and* $\mathtt{m}$ *be unary predicate symbols.*

(i) *Define* $\mathsf{RCVR}(\mathtt{m})$ *to be the* $\mathscr{L}^{\text{ring}}(\mathtt{m})$*-theory* RCVR *together with the sentence expressing that* $\mathtt{m}$ *is the set of non-units. Then* $\mathsf{RCVR}(\mathtt{m})$ *is complete, model complete and decidable.*

(ii) *Define* $\mathsf{RCVR}(\mathtt{b}, \mathtt{m})$ *to be the* $\mathscr{L}^{\text{ring}}(\mathtt{b}, \mathtt{m})$*-theory* $\mathsf{RCVR}(\mathtt{m})$ *together with the sentence expressing that* $\mathtt{b}$ *is a non-zero prime ideal properly contained in* $\mathtt{m}$*. Then* $\mathsf{RCVR}(\mathtt{b}, \mathtt{m})$ *is complete and model complete.*

Proof. (i) follows from [CD83, Theorems 4A and 4B] together with Lemma 2.3.11; (ii) follows from [Tre09, Corollary 6.3], since every model $(V, \mathfrak{b}, \mathfrak{m}_V) \models \mathsf{RCVR}(\mathtt{b}, \mathtt{m})$ is definable in $(\text{qf}(V), V, V_{\mathfrak{b}}) \models \mathsf{RCF}_{\text{convex},2}$. □

Lemma 2.3.14. *Let* V_1 *and* V_2 *be non-trivial real closed valuation rings regarded as* $\mathscr{L}^{\text{ring}}(\leq, \mathtt{m})$*-structures, where* $\leq$ *is a binary predicate interpreted as the total order relation and* $\mathtt{m}$ *is a unary predicate interpreted as the maximal ideal. If* A *is a totally* *ordered valuation ring such that $A \subseteq V_1, V_2$ as $\mathscr{L}^{\text{ring}}(\leq, \mathtt{m})$-structures, then there exists a non-trivial real closed valuation ring W which amalgamates V_1 and V_2 over A as $\mathscr{L}^{\text{ring}}(\leq, \mathtt{m})$-structures.*

Proof. By Lemma 2.3.11, $A \subseteq V_1, V_2$ as $\mathscr{L}^{\text{ring}}(\leq, \text{div})$-structures, therefore by the Cherlin-Dickmann theorem [CD83, Section 2] and [CK90, Proposition 3.5.19] there exists a non-trivial real closed valuation ring W amalgamating V_1 and V_2 over A as $\mathscr{L}^{\text{ring}}(\leq, \text{div})$-structures; conclude by appealing to Lemma 2.3.11 again. □

2.3.2 The ring of germs at a half-branch of a curve

This subsection gives a detailed description of a particular real closed valuation ring $\mathcal{O}_R$ which arises naturally in semi-algebraic geometry as the ring of germs of continuous semi-algebraic functions $X \longrightarrow R$ at a half-branch β of a semi-algebraic curve X over a real closed field R.

Definition 2.3.15 gives an alternate description of the ring $\mathcal{O}_R$, and this is followed by the identification of $\mathcal{O}_R$ with the ring of germs of continuous semi-algebraic functions $[0, 1] \longrightarrow R$ at 0^+, see Lemma 2.3.18. The subsection then continues with an analysis of half-branches of semi-algebraic curves over real closed fields (Definition 2.3.19 - Lemma 2.3.31), and it concludes showing in Proposition 2.3.35 that the definition of the ring $\mathcal{O}_R$ given in Definition 2.3.15 coincides with that in title of this subsection.

Throughout this subsection, R is a real closed field.

Any topological notions about a subset $X \subseteq R^m$ are always taken to be phrased with respect to the topology on X induced by the Euclidean topology on R^m ([BCR98, Definition 2.1.9]).

Definition 2.3.15. Let $R\langle t\rangle$ be the real closure of the function field $R(t)$ totally ordered by setting t to be a positive infinitesimal with respect to R. Define $\mathcal{O}_R$ to be the convex hull of R in $R\langle t\rangle$, that is

$$\mathcal{O}_R := \{s \in R\langle t\rangle \mid \exists r \in R \text{ such that } 0 \leq s \leq r\}.$$

$\mathcal{O}_R$ is a non-trivial real closed valuation ring by the implication (II) $\Rightarrow$ (I) in Theorem 2.3.6. Its maximal ideal $\mathfrak{m} := \mathfrak{m}_{\mathcal{O}_R}$ consists of those elements of $R\langle t\rangle$ which are infinitesimal with respect to R and $\mathcal{O}_R = R \oplus \mathfrak{m}$ as abelian groups; in particular, the map $R \longrightarrow \mathcal{O}_R/\mathfrak{m}$ given by $r \mapsto r/\mathfrak{m}$ is an isomorphism of real closed fields, see [DL95, Remark 2.11]

Before identifying $\mathcal{O}_R$ with the ring of germs of continuous semi-algebraic functions $[0,1] \longrightarrow R$ at 0^+ in Lemma 2.3.18, a model theoretic fact (Lemma 2.3.16) and a semi-algebraic result (Theorem 2.3.17) are needed.

Lemma 2.3.16. *Let $R\langle t\rangle$ be as in* Definition 2.3.15*, and let $\varphi(x)$ be any $\mathscr{L}^{\mathrm{poring}}(R)$-formula. The following are equivalent:*

(i) *There exists $\varepsilon \in R$ with $\varepsilon > 0$ such that $[0,\varepsilon]_R \subseteq \varphi(R) := \{r \in R \mid R \models \varphi(r)\}$.*

(ii) *$R\langle t\rangle \models \varphi(t)$, that is, $\varphi(x) \in \mathrm{tp}(t/R)$.*

Proof. (i) $\Rightarrow$ (ii). Item (i) is equivalent to the statement that $R \models \forall x(0 \le x \le \varepsilon \rightarrow \varphi(x))$, from which (ii) follows.

(ii) $\Rightarrow$ (i). It suffices to show that

$$R \models \exists y \forall x[(y > 0 \curlywedge 0 \le x \le y) \rightarrow \varphi(x)]. \tag{$*$}$$

Assume for contradiction that $(*)$ does not hold. Then the set of $\mathscr{L}^{\mathrm{poring}}(R)$-formulas

$$\Sigma(x) := \{(r > 0 \curlywedge 0 \le x \le r) \curlywedge \neg\varphi(x) \mid r \in R\}$$

is finitely consistent, therefore by (model-theoretic) compactness $\Sigma(x)$ has a realization s in an elementary extension $S \succeq R$. In particular, s and t determine the same cut in R, therefore $\mathrm{tp}(t/R) = \mathrm{tp}(s/R)$ by o-minimality (see [PS86, Theorem 3.3]), and thus $S \models \varphi(s)$ by (ii), yielding the required contradiction. □

Theorem 2.3.17 (Semi-algebraic Tietze extension theorem)**.** *Let R be a real closed field. Let D be a locally closed*[10] *semi-algebraic set in R^m and $C \subseteq D$ be semi-algebraic and closed in D. If $f : C \longrightarrow R$ is a continuous semi-algebraic function, then there exists a continuous semi-algebraic function $\widehat{f} : D \longrightarrow R$ such that $\widehat{f}_{\restriction C} = f$.*

Proof. See [BCR98, Proposition 2.6.9] or [Dri98, Chapter 8, Corollary 3.10]. □

[10] A semi-algebraic set is *locally closed* if it is the intersection of a closed semi-algebraic set and an open semi-algebraic set.

Lemma 2.3.18. *Let R be a real closed field. Consider the composite R-algebra homomorphism*

$$\begin{array}{rcccl} \Phi : C_{\text{s.a.}}([0,1]_R) & \hookrightarrow & C_{\text{s.a.}}([0,1]_{R\langle t\rangle}) & \twoheadrightarrow & R\langle t\rangle \\ f & \longmapsto & f_{R\langle t\rangle} & \longmapsto & f_{R\langle t\rangle}(t), \end{array}$$

where:

(i) $[0,1]_R$ *is the unit interval in* R*, and similarly for* $[0,1]_{R\langle t\rangle}$*,*

(ii) $C_{\text{s.a.}}([0,1]_R)$ *is the ring of continuous semi-algebraic functions* $[0,1]_R \longrightarrow R$*, and similarly for* $C_{\text{s.a.}}([0,1]_{R\langle t\rangle})$*.*

(iii) $f_{R\langle t\rangle}$ *is the function* $[0,1]_{R\langle t\rangle} \longrightarrow R\langle t\rangle$ *defined by the same* $\mathscr{L}^{\text{poring}}(R)$*-formula which defines* $f \in C_{\text{s.a.}}([0,1]_R)$*.*

Then $\text{im}(\Phi) = \mathcal{O}_R$, $\ker(\Phi) = \{f \in C_{\text{s.a.}}([0,1]_R) \mid \exists\varepsilon > 0 \text{ such that } f_{\restriction[0,\varepsilon]} = 0\}$*, and* $\Phi(f)/\mathfrak{m} = f(0)/\mathfrak{m}$ *for all* $f \in C_{\text{s.a.}}([0,1]_R)$*.*

Proof. Pick $f \in C_{\text{s.a.}}([0,1]_R)$ such that $\Phi(f) \geq 0$. By [BCR98, Proposition 2.6.2] there exists a polynomial $p(x) \in R[x]$ such that $|f(x)| \leq p(x)$ for all $x \in [0,1]_R$, therefore also $|f_{R\langle t\rangle}(x)| \leq p(x)$ for all $x \in [0,1]_{R\langle t\rangle}$. Since $p(t) \in \mathcal{O}_R$, it follows from convexity of $\mathcal{O}_R$ in $R\langle t\rangle$ that $f_{R\langle t\rangle}(t) = \Phi(f) \in \mathcal{O}_R$, therefore $\text{im}(\Phi) \subseteq \mathcal{O}_R$.

To show that $\mathcal{O}_R \subseteq \text{im}(\Phi)$, pick $s \in \mathcal{O}_R$ and assume without loss of generality that $s \geq 0$. By standard arguments, the real closed field $R\langle t\rangle$ is exactly the definable closure of $R \cup \{t\}$ in $R\langle t\rangle$, therefore there exists a semi-algebraic function $g : R \longrightarrow R$ such that $g_{R\langle t\rangle}(t) = s$, see for example [DL95, Sections 2 and 3] and the references therein. By the Monotonicity theorem (see [Dri98, Chapter 3, Section 1]), there exist $a_1, \dots, a_m \in R$ such that $a_1 < \cdots < a_m$ and g is continuous on each interval $(-\infty, a_1), \dots, (a_i, a_{i+1}), \dots, (a_m, \infty)$. There are two possible cases:

<u>Case 1:</u> $0 \neq a_i$ for all $i \in [m]$. Then 0 belongs to an open interval on which g is continuous, from which it follows that there exists $\varepsilon > 0$ such that g is continuous on $[0, \varepsilon]$. By Theorem 2.3.17 there exists $f \in C_{\text{s.a.}}([0,1]_R)$ such that $f_{\restriction[0,\varepsilon]} = g_{\restriction[0,\varepsilon]}$, and $\Phi(f) = f_{R\langle t\rangle}(t) = g_{R\langle t\rangle}(t) = s$ for such f.

<u>Case 2:</u> There exists $i \in [m]$ such that $0 = a_i$. By choice of s, there exists $r \in R$ such that $g(t) = s \leq r$, therefore by Lemma 2.3.16 there exists $\varepsilon > 0$ such that g is

R-bounded on $[0, \varepsilon] = [a_i, \varepsilon]$. In particular, $\lim_{x \to 0^+} g(x)$ exists in R, and assuming without loss of generality that $\varepsilon \in (a_i, a_{i+1})$, it follows that g is continuous on $[0, \varepsilon]$. As in Case 1, any $f \in C_{\text{s.a.}}([0,1]_R)$ extending $g_{\restriction [0,\varepsilon]}$ satisfies $\Phi(f) = s$.

The fact that $\ker(\Phi) = \{f \in C_{\text{s.a.}}([0,1]_R) \mid \exists \varepsilon > 0 \text{ such that } f_{\restriction[0,\varepsilon]} = 0\}$ follows easily from Lemma 2.3.16. Finally, pick $f \in C_{\text{s.a.}}([0,1]_R)$. By choice of t, the extension of real closed fields $R \subseteq R\langle t \rangle$ is tame in the sense of [DL95], therefore every $s \in \mathcal{O}_R$ has a standard part, that is, a unique element $\operatorname{st}(s) \in R$ such that $|s - \operatorname{st}(s)| < \varepsilon$ for all $\varepsilon \in R$ with $\varepsilon > 0$. In fact, $\operatorname{st}(s) \in R$ is the unique element in R such that $\operatorname{st}(s)/\mathfrak{m} = s/\mathfrak{m}$ (see [DL95, Remark 2.11]), therefore since $\Phi(f) \in \mathcal{O}_R$ it follows that

$$\Phi(f)/\mathfrak{m} = f_{R\langle t\rangle}(t)/\mathfrak{m} = \operatorname{st}(f_{R\langle t\rangle}(t))/\mathfrak{m} \overset{(*)}{=} f(\operatorname{st}(t))/\mathfrak{m} = f(0)/\mathfrak{m}$$

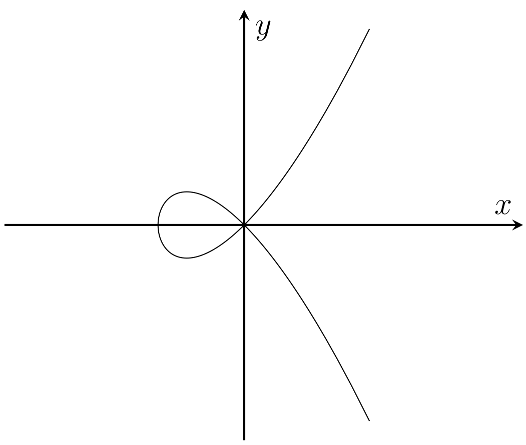
where $(*)$ is by the non-standard characterization of continuity, see for instance [DL95, Lemma 1.13]. ☐

Definition 2.3.19. A (*semi-algebraic*) *curve* is a 1-dimensional semi-algebraic subset $X \subseteq R^m$ without isolated points.

Example 2.3.20. (i) Every *(affine) real algebraic curve* (that is, a 1-dimensional algebraic subset of R^m for some $m \in \mathbb{N}$) without isolated points[11] is a semi-algebraic curve, for instance, the set of solutions in $\mathbb{R}^2$ of the equation

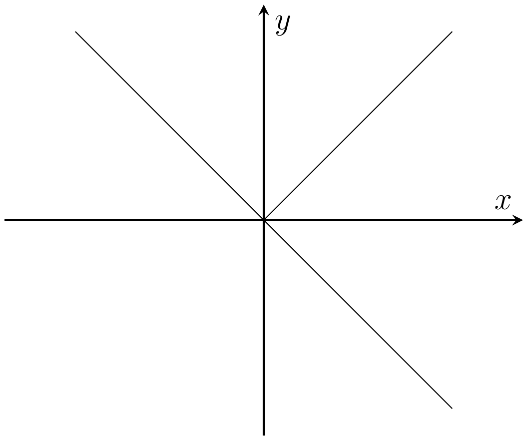
$$x^2 + x^3 - y^2 = 0$$

is a real algebraic curve:



[11] In general, real algebraic curves can have isolated points: for example, the set of solutions in $\mathbb{R}^2$ of the equation $x^2 + y^2 - x^3 = 0$ is a real algebraic curve having $(0, 0)$ as unique isolated point.

(ii) Not every semi-algebraic curve is a real algebraic curve. For example, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by $f(x) := -x$ and let $\Gamma(f) \subseteq \mathbb{R}^2$ be its graph; the set $X := \Gamma(f) \cup \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } x = y\}$ a semi-algebraic curve:



Then X is not a real algebraic curve since the number of *half-branches* (Definition 2.3.26) of a real algebraic curve at a point is always even ([BCR98, Theorem 9.5.7]) and X has exactly three half-branches at $(0, 0) \in X$.

Intuitively, a half-branch of a semi-algebraic curve X at a point $a \in X$ captures a direction from which one can get arbitrarily close to a within X; for instance, the real algebraic curve in Example 2.3.20 (i) has exactly four half-branches at $(0, 0)$, and every point in $X := R$ has exactly two half-branches. Half-branches are formally defined as being a particular kind of germs of semi-algebraic sets (Definition 2.3.21), see Definition 2.3.26 as well as Lemma 2.3.28.

Throughout the rest of this subsection, $X \subseteq R^m$ is a semi-algebraic curve.

Definition 2.3.21. Let $Y_1, Y_2 \subseteq X$ be semi-algebraic and let $a \in X$. Say that Y_1 and Y_2 *have the same germ at* a if there exists a semi-algebraic neighbourhood U of a such that $Y_1 \cap U = Y_2 \cap U$. This defines an equivalence relation on the set of semi-algebraic subsets of X. A *germ (of semi-algebraic subsets)* γ *of* X *(centred) at* a is an equivalence class of semi-algebraic subsets $Y \subseteq X$ which have the same germ at a.

Example 2.3.22. Let $a \in X$.

(i) The set of all $Y \subseteq X$ semi-algebraic such that a is isolated in Y is a germ of X at a.

(ii) The set of all $Y \subseteq X$ semi-algebraic such that $a \notin \mathrm{cl}_X(Y)$ is a germ of X at a, where $\mathrm{cl}_X(Y)$ is the closure of Y in X.

(iii) If γ is a germ of X at a, then either $a \in Y$ for all $Y \in \gamma$ or $a \notin Y$ for all $Y \in \gamma$. In particular, if γ satisfies the property that $a \notin Y$ for all $Y \in \gamma$, then $\gamma' := \{Y \cup \{a\} \mid Y \in \gamma\}$ is a germ of X at a distinct from γ. For instance, if $X := R$ and $a := 0$, then the intervals $(0, 1), [0, 1) \subseteq X$ are representatives of two distinct germs of X at a.

Definition 2.3.23. A germ of X at a is *non-degenerate* if it is neither of the germs described in items (i) and (ii) in Example 2.3.22.

Lemma 2.3.24. *A germ γ of X at a is non-degenerate if and only if for all $Y \in \gamma$ and all semi-algebraic neighbourhoods U of a, $\mathrm{int}_X(Y \cap U) \neq \varnothing$, where int$(Y \cap U)$ is the interior of $Y \cap U$ in X.*

Proof. Suppose first that γ is non-degenerate. Let $Y \in \gamma$ and let U be a semi-algebraic neighbourhood of a; then exactly one of the following cases holds:

(i) $a \in Y$. In this case $\mathrm{int}_X(Y \cap U') \neq \varnothing$ for any open semi-algebraic neighbourhood $U' \subseteq X$ of a; otherwise, by o-minimality, $\mathrm{int}_X(Y \cap U') = \varnothing$ implies that $Y \cap U'$ is a finite set of points such that $a \in Y \cap U'$, thus yielding the existence of an open neighbourhood $U'' \subseteq U'$ of a with $\{a\} = Y \cap U''$, a contradiction to γ being non-degenerate. Since U is a semi-algebraic neighbourhood of a, there exists an open semi-algebraic subset $U' \subseteq U$ such that $a \in U'$, therefore $\varnothing \neq \mathrm{int}_X(Y \cap U') \subseteq \mathrm{int}_X(Y \cap U)$, as required.

(ii) $a \in \mathrm{cl}_X(Y) \setminus Y$. In this case $a \notin Y \cap U$ and $a \in \partial_X(Y)$, where $\partial_X(Y)$ is the boundary of Y in X. Assume for contradiction that $\mathrm{int}_X(Y \cap U) = \varnothing$; then $Y \cap U$ is a finite set of points by o-minimality, therefore there exists a semi-algebraic neighbourhood U' of a such that $U' \subseteq U$ and $Y \cap U' = \varnothing$, a contradiction to $a \in \partial_X(Y)$.

Conversely let $Y \in \gamma$; it will be shown that the condition in the lemma is not satisfied if γ is one of the germs in items (i) and (ii) in Example 2.3.22:

(i) $a \in Y$ and a is isolated in Y. In this case there exists an open neighbourhood U of a such that $Y \cap U = \{a\}$, and since X has no isolated points, $\{a\}$ is not open in X, therefore $\mathrm{int}_X(Y \cap U) = \varnothing$.

(ii) $a \notin \mathrm{cl}(Y)$. In this case $a \in \mathrm{int}_X(X \setminus Y)$, therefore there exists an open neighbourhood U of a such that $Y \cap U = \varnothing$, and thus $\mathrm{int}_X(Y \cap U) = \varnothing$. □

Recall from [BCR98, Definition 2.4.2] or [Dri98, p. 19, Definition 3.5] that $Y \subseteq X$ is *semi-algebraically connected* if for all disjoint semi-algebraic subsets $Y_1, Y_2 \subseteq Y$ which are closed in Y, if $Y = Y_1 \cup Y_2$, then either $Y = Y_1$ or $Y = Y_2$; in particular, $Y := \varnothing$ is semi-algebraically connected. Moreover, Y is semi-algebraically connected if and only if Y is *semi-algebraically path connected*, see [BCR98, Proposition 2.5.13]; in fact:

Remark 2.3.25. Let $Y \subseteq X$ be semi-algebraic with non-empty interior; then $Y \subseteq X$ is semi-algebraically connected if and only if Y is *semi-algebraically arc-connected*, that is, if and only if for all distinct $a, b \in Y$ there exists a continuous semi-algebraic map $\sigma : [0, 1] \longrightarrow X$ such that $\sigma(0) = a$ and $\sigma(1) = b$ which is a homeomorphism onto its image. This actually holds for any Y with non-empty interior which is definable and definably connected in an o-minimal expansion of an abelian o-group: one can easily check that the proof of [Dri98, Chapter 6, Proposition 3.2] can be used *mutatis mutandis* to show that any two distinct points in such Y can be connected by an injective definable path, and such a path is a homeomorphism onto its image by [Dri98, Chapter 6, Corollary 1.12].

Definition 2.3.26. A *half-branch of X (centred) at a* is a non-degenerate germ β of X at a such that for any $Y \in \beta$, $a \in Y$ and there exists a semi-algebraic neighbourhood U of a such that $Y \cap (U \setminus \{a\})$ is semi-algebraically connected. A *half-branch of X* is a half-branch of X at a for some $a \in X$.

Remark 2.3.27. (i) The condition that $a \in Y$ for all $Y \in \beta$ in Definition 2.3.26 is needed to avoid "double counting" of half-branches: if this condition were not included, then $(0, 1)$ and $[0, 1)$ would be representatives of two distinct half-branches of $X := R$ at $a := 0$, see Example 2.3.22 (iii). Note also that by Example 2.3.22 (iii) one could replace the condition that $a \in Y$ for all $Y \in \beta$ in Definition 2.3.26 by $a \notin Y$ for all $Y \in \beta$ in order to define what a half-branch is.

(ii) In [Phi15, Definition 2.3.1.2], neither the non-degeneracy condition nor the condition discussed in item (i) above is included in Phillips' definition of a *branch*. Every half-branch in the sense of Definition 2.3.26 is a branch in the sense of Phillips, but the converse is not true: in particular, the germ in Example 2.3.22 (i) is a branch in the sense of Phillips, and $(0,1)$ and $[0,1)$ are representatives of two distinct branches of $X := R$ at $a := 0$ in the sense of Phillips.

(iii) If X is furthermore a real algebraic curve, then it follows from Example 2.3.22 (iii) that half-branches β of X at a in the sense of Definition 2.3.26 are in bijective correspondence with half-branches of X centred at a in the sense of [BCR98, Definition 9.5.2]; in particular a germ γ of X at a is a half-branch in the sense of Definition 2.3.26 if and only if $\{Y \setminus \{a\} \mid Y \in \gamma\}$ is a half-branch of X centred at a in the sense of [BCR98, Definition 9.5.2].

Lemma 2.3.28 (cf. Proposition 9.5.1 in [BCR98])**.** *A germ γ of X at a is a half-branch if and only if for every $C \in \gamma$ there exists $Y \in \gamma$ with $Y \subseteq C$ together with a semi-algebraic homeomorphism $\sigma : [0,1]_R \longrightarrow Y$ such that $\sigma(0) = a$, where $[0,1]_R$ is the closed unit interval in R.*

Proof. Suppose first that γ is a half-branch and let $C \in \gamma$. Pick a semi-algebraic neighbourhood U of a such that $C \cap (U \setminus \{a\})$ is semi-algebraically connected; then $Y := C \cap U \in \gamma$ is a semi-algebraically path connected semi-algebraic subset of X with non-empty interior (Lemma 2.3.24), therefore, by replacing U with a semi-algebraic neighbourhood U' of a such that $U' \subsetneq U$ if necessary, Y is semi-algebraically homeomorphic to $[0,1]$ via a map σ as in the statement of the lemma. The converse is straightforward from the definition of half-branch. □

Definition 2.3.29. (i) A *curve interval C of X at a* is a semi-algebraic subset $C \subseteq X$ for which there exists a semi-algebraic homeomorphism $\sigma : [0,1] \longrightarrow C$ such that $\sigma(0) = a$; a *curve interval of X* is a curve interval of X at a for some $a \in X$.

(ii) Let β be a half-branch of X at a. A semi-algebraic subset $C \subseteq X$ is a *curve interval of β* if C is a curve interval at a such that $C \in \beta$ (such C always exists by Lemma 2.3.28).

Example 2.3.30. Let $a \in X$ and suppose that X has $n \in \mathbb{N}$ half-branches β_i of X at a. For each $\varepsilon > 0$, let $\overline{B}_\varepsilon(a)$ be the closed ball in X of radius ε around a. Then for small enough $\varepsilon > 0$, the set $\overline{B}_\varepsilon(a) \setminus \{a\}$ has n semi-algebraically connected components U, and each $\mathrm{cl}_X(U) = U \cup \{a\}$ is a curve interval of some half-branch β_i.

Lemma 2.3.31. *Let $C \subseteq X$ be closed and semi-algebraic. The following are equivalent:*

(i) *C is a curve interval (i.e., C is semi-algebraically homeomorphic to $[0, 1]$).*

(ii) *C is semi-algebraically connected and there exist distinct $a, b \in C$ such that C is minimal (under subset inclusion) amongst closed, semi-algebraic, and semi-algebraically connected $D \subseteq X$ with $a, b \in D$.*

Proof. The proof of the implication (i) $\Rightarrow$ (ii) is straightforward, so suppose that (ii) holds and let $a, b \in C$ witness this; since $a, b \in C$ are distinct and C is semi-algebraically connected, C must have non-empty interior, therefore by Remark 2.3.25 there exists a semi-algebraic map $\sigma : [0, 1] \longrightarrow C$ such that $\sigma(0) = a$ and $\sigma(1) = b$ which is a homeomorphism onto its image; such σ is surjective by the minimality condition on C and by [Dri98, Chapter 6, Corollary 1.12], therefore (i) follows. □

Definition 2.3.32. Let $a \in X$. The *branching degree of a* is the number of half-branches of X at a.

Note that by o-minimality the branching degree of each $a \in X$ is finite, and since it is assumed that X does not have any isolated points, each $a \in X$ has branching degree at least 1.

To every half-branch β of X one can associate two objects:

Definition 2.3.33. Let β be a half-branch of X. Define

$$\mathfrak{p}_\beta := \{f \in C_{\text{s.a.}}(X) \mid \exists Y \in \beta \text{ such that } f_{\restriction Y} = 0\}$$

and

$$\mathfrak{f}_\beta := \{C \subseteq X \mid C \text{ is closed and semi-algebraic, and } C \in \beta\}.$$

Clearly $\mathfrak{p}_\beta$ is an in ideal of $C_{\text{s.a.}}(X)$, and the elements in $C_{\text{s.a.}}(X)/\mathfrak{p}_\beta$ are the germs of functions in $C_{\text{s.a.}}(X)$ at the half-branch β. The relationship between $\mathfrak{p}_\beta$, $\mathfrak{f}_\beta$ and $\mathcal{O}_R$

is given in Proposition 2.3.35. First is needed an equivalent description of the lattice of closed and semi-algebraic subsets of X:

Lemma 2.3.34. *For each $f \in C_{\text{s.a.}}(X)$, let $\{f = 0\} := \{x \in X \mid f(x) = 0\}$ be its zero set. The set $L_X := \{\{f = 0\} \mid f \in C_{\text{s.a.}}(X)\}$ is exactly the lattice of closed and semi-algebraic subsets of X.*

Proof. This follows from the fact that any closed and semi-algebraic $S \subseteq R^m$ is the zero set of its distance function, which is continuous and semi-algebraic, see [BCR98, Proposition 2.2.8]. Let $C \subseteq X$ closed and semi-algebraic; the proof that follows shows an alternative construction of an $f \in C_{\text{s.a.}}(X)$ such that $C = \{f = 0\}$. Consider the following cases:

Case 1: $\partial_X(C) = \varnothing$. In this case C is clopen in X (see for example [Men62, p 103, Exercise 7]), therefore the map $f : X \longrightarrow R$ given by $f(x) := 0$ if $x \in C$ and $f(x) := 1$ if $x \in X \setminus C$ is continuous and semi-algebraic, and $C = \{f = 0\}$.

Case 2: $\partial_X(C) \neq \varnothing$. By o-minimality and since X is 1-dimensional, $\partial_X(C) := \{a_1, \dots, a_m\}$ for some $n \in \mathbb{N}$. For each $i \in [m]$, let $\beta_{i1}, \dots, \beta_{in_i}$ be all the half-branches of X at a_i, where n_i is the branching degree of a_i. Choose $\varepsilon > 0$ small enough such that for each $i \in [m]$ the following conditions hold:

1. For all $i \in [m]$, the set $\overline{B}_\varepsilon(a_i) \setminus \{a_i\}$ (see Example 2.3.30) has n_i connected components $U_{i1}, \dots, U_{in_i}$ and $C_{ij} := \text{cl}_X(U_{ij}) = U_{ij} \cup \{a_i\}$ is a curve interval of β_{ij} for all $j \in [n_i]$.

2. $C_{ij_1} \cap C_{ij_2} = \{a_i\}$ for all $i \in [m]$ and all $j_1, j_2 \in [n_i]$ with $j_1 \neq j_2$.

3. $C_{i_1j_1} \cap C_{i_2j_2} = \varnothing$ for all $i_1, i_2 \in [m]$ such that $i_1 \neq i_2$ and all $j_1 \in [n_{i_1}]$ and $j_2 \in [n_{i_2}]$.

4. For all $i \in [m]$ and all $j \in [n_i]$, either $C_{ij} \subseteq C$ or $C_{ij} \cap C = \{a_i\}$.

For each $i \in [m]$ and each $j \in [n_i]$, let $\sigma_{ij} : [0,1]_R \longrightarrow C_{ij}$ be a semi-algebraic homeomorphism such that $\sigma_{ij}(0) = a_i$. Let $D := C \cup \bigcup_{i\in[m]} \bigcup_{j\in[n_i]} C_{ij}$ and define the map $D \longrightarrow R$ by

$$f_0(x) := \begin{cases} 0 & \text{if } x \in C \\ \sigma_{ij}^{-1}(x) & \text{if } x \in C_{ij} \text{ and } C_{ij} \cap C = \{a_i\}. \end{cases}$$

f_0 is a well-defined continuous semi-algebraic function $D \longrightarrow R$ and $f(b) = 1$ for every $b \in \partial_X(D)$ by construction. In particular, the map $f : X \longrightarrow R$ defined by $f(x) := f_0(x)$ if $x \in D$ and $f(x) := 1$ if $x \in \mathrm{cl}_X(X \setminus D)$ is continuous and semi-algebraic, and $C = \{f = 0\}$ by construction, as required. □

Proposition 2.3.35. *Let β be a half-branch of X centred at $a \in X$.*

(i) *$\mathfrak{p}_\beta$ is a prime ideal of A and the ring of germs $C_{\text{s.a.}}(X)/\mathfrak{p}_\beta$ is canonically isomorphic to $\mathcal{O}_R$ (see* Definition 2.3.15*). In particular, let C be any curve interval of β; then $\mathfrak{p}_\beta = \ker(F_C)$, where F_C is the composite R-algebra homomorphism*

$$\begin{array}{rcccccl} F_C : C_{\text{s.a.}}(X) & \twoheadrightarrow & C_{\text{s.a.}}(C) & \longrightarrow & C_{\text{s.a.}}([0,1]_R) & \twoheadrightarrow & \mathcal{O}_R \\ f & \longmapsto & f_{\restriction C} & \longmapsto & f_{\restriction C} \circ \sigma & \longmapsto & \Phi(f_{\restriction C} \circ \sigma), \end{array}$$

$\sigma : [0,1]_R \longrightarrow C$ is any semi-algebraic homeomorphism such that $\sigma(0) = a$, and $\Phi : C_{\text{s.a.}}([0,1]_R) \twoheadrightarrow \mathcal{O}_R$ is the ring homomorphism defined in Lemma 2.3.18.

(ii) *Let $[-]_\beta : C_{\text{s.a.}}(X) \twoheadrightarrow \mathcal{O}_R$ be the canonical surjection of item* (i)*, that is,*

$$[f]_\beta := f/\mathfrak{p}_\beta = \Phi(f_{\restriction C} \circ \sigma)$$

for any curve interval C of β and any semi-algebraic homeomorphism $\sigma : [0,1]_R \longrightarrow C$ such that $\sigma(0) = a$. Then $[f]_\beta/\mathfrak{m} = f(a)/\mathfrak{m}$ for every $f \in C_{\text{s.a.}}(X)$. In particular, the following are equivalent for all $f, g \in C_{\text{s.a.}}(X)$, where $\overset{=}{\leq}$ is either $=$ or $\leq$:

(a) $[f]_\beta/\mathfrak{m} \overset{=}{\leq} [g]_\beta/\mathfrak{m}$.

(b) $f(a) \overset{=}{\leq} g(a)$.

(iii) *$\mathfrak{f}_\beta$ is a prime filter in the lattice $L_X := \{\{f = 0\} \mid f \in C_{\text{s.a.}}(X)\}$ of closed semi-algebraic subsets of X (see* Lemma 2.3.34*).*

(iv) *Let $f, g \in C_{\text{s.a.}}(X)$ and set $\{f \overset{=}{\leq} g\} := \{x \in X \mid f(x) \overset{=}{\leq} g(x)\}$, where $\overset{=}{\leq}$ is either $=$ or $\leq$. The following are equivalent:*

(a) $[f]_\beta \overset{=}{\leq} [g]_\beta$.

(b) $\{f \overset{=}{\leq} g\} \in \mathfrak{f}_\beta$.

(c) *There exists a curve interval C of β such that $C \subseteq \{f \mathrel{\substack{=\\ \leq}} g\}$.*

Proof. (i). It suffices to show that $\mathfrak{p}_\beta = \ker(F_C)$, where C is any curve interval of β and F_C is defined as in item (i). For one inclusion one has

$$\begin{aligned}\ker(F_C) &= \{f \in C_{\text{s.a.}}(X) \mid \Phi(f_{\restriction C} \circ \sigma) = 0\} \\ &= \{f \in C_{\text{s.a.}}(X) \mid f_{\restriction C} \circ \sigma \in \ker(\Phi)\} \\ &\overset{(1)}{=} \{f \in C_{\text{s.a.}}(X) \mid \exists \varepsilon > 0 \text{ such that } (f_{\restriction C} \circ \sigma)_{\restriction [0,\varepsilon]} = 0\} \\ &= \{f \in C_{\text{s.a.}}(X) \mid \exists \varepsilon > 0 \text{ such that } f_{\restriction \sigma([0,\varepsilon])} = 0\} \\ &\overset{(2)}{\subseteq} \mathfrak{p}_\beta,\end{aligned}$$

where (1) follows by Lemma 2.3.18 and (2) follows since $\sigma([0,\varepsilon]) \in \beta$. For the other inclusion, pick $f \in C_{\text{s.a.}}(X)$ and $Y \in \beta$ such that $f_{\restriction Y} = 0$. Then $C \cap Y \in \beta$, therefore by Lemma 2.3.28 there exists a curve interval Y' of β such that $Y' \subseteq Y \cap C$; such curve interval Y' is of the form $\sigma([0,\varepsilon'])$ for some $\varepsilon' \in (0,\varepsilon)$, therefore $f \in \ker(F_C)$ follows by the chain of equalities above.

(ii). Let f, C, and σ be as in item (ii). Then $[f]_\beta/\mathfrak{m} = \Phi(f_{\restriction C} \circ \sigma)/\mathfrak{m} = (f_{\restriction C} \circ \sigma)(0)/\mathfrak{m} = f(\sigma(0))/\mathfrak{m} = f(a)/\mathfrak{m}$ by Lemma 2.3.18. The equivalence of (ii) (a) and (ii) (b) follows thus immediately from the fact that $r \mapsto r/\mathfrak{m}$ is an isomorphism of real closed fields $R \longrightarrow \mathcal{O}_R/\mathfrak{m}$, see the discussion after Definition 2.3.15.

(iii). That $\mathfrak{f}_\beta$ is a filter in L_X is clear by the definition of half-branch given in Definition 2.3.26. To show that $\mathfrak{f}_\beta$ is prime, suppose that $C \cup D \in \mathfrak{f}_\beta$, so that there exist $f, g \in C_{\text{s.a.}}(X)$ such that $C = \{f = 0\}$, $D = \{g = 0\}$, and $C \cup D = \{fg = 0\} \in \beta$. By Lemma 2.3.28, there exists a curve interval Y of β such that $Y \subseteq \{fg = 0\}$, that is, such that $(fg)_{\restriction Y} = 0$, therefore $fg \in \mathfrak{p}_\beta$. It follows by (i) that either $f \in \mathfrak{p}_\beta$ or $g \in \mathfrak{p}_\beta$, therefore either there exists $Y_1 \in \beta$ with $Y_1 \subseteq \{f = 0\} = C$ (and thus $C \in \beta$), or there exists $Y_2 \in \beta$ with $Y_2 \subseteq \{g = 0\} = D$ (and thus $D \in \beta$).

(iv). The equivalence of (b) and (c) is clear by Lemma 2.3.28, so it suffices to prove the equivalence of (a) and (c). Note also that for all $f, g \in C_{\text{s.a.}}(X)$,

$$[f]_\beta \leq [g]_\beta \iff \min\{[f]_\beta, [g]_\beta\} = [f]_\beta \overset{(*)}{=} [f \wedge g]_\beta = [f]_\beta,$$

where $\wedge$ is the meet operation in the real closed ring $C_{\text{s.a.}}(X)$ (more precisely, $f \wedge g \in C_{\text{s.a.}}(X)$ is given by $(f \wedge g)(x) := \min\{f(x), g(x)\}$) and $(*)$ follows from Theorem 2.3.2

(III), therefore it suffices to consider the case where $\overset{=}{\leq}$ is $=$. Then

$$[f]_\beta = [g]_\beta \iff [f-g]_\beta = 0 \iff f-g \in \mathfrak{p}_\beta \iff \exists Y \in \beta \text{ such that } Y \subseteq \{f=g\},$$

and the statement on the right hand side of the equivalences is in turn equivalent to (c) by Lemma 2.3.28, concluding thus the proof. □

Example 2.3.36. Each $a \in X := R$ has exactly two half-branches, denoted by a^- and a^+. The sets $[a-1,a]$ and $[a,a+1]$ are curve intervals of a^- and a^+, respectively, therefore $\mathfrak{p}_{a^-} := \{f \in C_{\text{s.a.}}(R) \mid \exists \varepsilon \in (0,1) \text{ such that } f_{\restriction [a-\varepsilon,a]} = 0\}$ and $\mathfrak{p}_{a^+} := \{f \in C_{\text{s.a.}}(R) \mid \exists \varepsilon \in (0,1) \text{ such that } f_{\restriction [a,a+\varepsilon]} = 0\}$ by Proposition 2.3.35 (i), and if $\mathfrak{m}_a := \{f \in C_{\text{s.a.}}(R) \mid f(a) = 0\}$, then $\mathfrak{m}_a/\mathfrak{p}_{a^\pm}$ is the maximal ideal $\mathfrak{m}$ in $\mathcal{O}_R \cong C_{\text{s.a.}}(R)/\mathfrak{p}_{a^\pm}$ by Proposition 2.3.35 (ii).

Corollary 2.3.37. *Let $a \in X$ and let $\beta_1, \dots, \beta_n$ be all the half-branches of X at a. The ring of germs of $C_{\text{s.a.}}(X)$ at a (that is, the localization of $C_{\text{s.a.}}(X)$ at the maximal ideal $\mathfrak{m}_a := \{f \in C_{\text{s.a.}}(X) \mid f(a) = 0\}$) is canonically isomorphic to the n-fold fibre product*

$$(C_{\text{s.a.}}(X)/\mathfrak{p}_{\beta_1} \times_R C_{\text{s.a.}}(X)/\mathfrak{p}_{\beta_2}) \times_R \dots) \times_R C_{\text{s.a.}}(X)/\mathfrak{p}_{\beta_n}). \qquad (*)$$

Proof. Consider the canonical map $C_{\text{s.a.}}(X)_{\mathfrak{m}_a} \longrightarrow \prod_{i=1}^n C_{\text{s.a.}}(X)/\mathfrak{p}_{\beta_i}$ sending each germ of a function $f \in C_{\text{s.a.}}(X)$ at a to the tuple in $\prod_{i=1}^n C_{\text{s.a.}}(X)/\mathfrak{p}_{\beta_i}$ whose coordinates are the germs of f at each of the half-branches β_i of X at a. This map is an injective ring homomorphism with image $(*)$. □

2.4 Lattice-ordered abelian groups

In this section the theory of lattice-ordered abelian groups (ℓ-groups for short) is presented as needed in order to develop the model-theoretic machinery on which Chapter 3 builds upon, namely the two-sorted model-theoretic analysis of lattice-ordered abelian groups of functions as developed by Shen and Weispfenning in [SW87a], see Subsection 2.4.3. In particular, the focus in Subsections 2.4.1 and 2.4.2 is to give functional representations of arbitrary lattice-ordered abelian groups (see Corollary 2.4.7) and to relate these functional representations with some basic structural properties of ℓ-groups in order to analyze their model-theory via the Shen-Weispfenning theorem.

Standard references for the theory of ℓ-groups are [Dar95], [BKW77], and [AF88]. The material present in this section, and in particular that in Subsection 2.4.3, follows largely the exposition given in [Tre22] and [Tre].

2.4.1 Preliminaries on ℓ-groups

Definition 2.4.1. (i) A *lattice-ordered abelian group* (abbreviated as *ℓ-group*) is an abelian group $(G, +, -, 0)$ together with a partial order $\leq$ such that $(G, \leq)$ is a lattice and such that

$$f \leq g \implies f + h \leq g + h$$

for all $f, g, h \in G$. An *o-group* is an ℓ-group which is totally ordered.

(ii) A function $f : G \longrightarrow H$ between ℓ-groups G and H is an *ℓ-homomorphism* if it is both a group homomorphism and a lattice homomorphism; if f is moreover injective it is an *ℓ-group embedding*.

(iii) Write ℓ-**Gp** for the category of ℓ-groups together with ℓ-homomorphisms.

Direct products of o-groups are ℓ-groups under componentwise group and lattice operations, therefore the set of all functions $X \longrightarrow N$ from a non-empty set X to an o-group N is an ℓ-group; in fact, every ℓ-group is isomorphic to an ℓ-subgroup of such ℓ-group of functions, see Corollary 2.4.7. Note also that Note that the additive group of any real closed ring (Definition 2.3.1) is an ℓ-group.

Throughout the remaining part of this subsection, G is an ℓ-group.

Definition 2.4.2. (i) An *ℓ-ideal* $I \subseteq G$ is a convex ℓ-subgroup of G. If $S \subseteq G$ is a non-empty set, define

$$\ell(S) := \bigcap\{I \subseteq G \mid I \text{ is an } \ell\text{-ideal and } S \subseteq I\}$$

to be ℓ-ideal generated by S. If $S = \{f\}$ for some $f \in G$, set $\ell(f) := \ell(\{f\})$; ℓ-ideals of the form $\ell(f)$ for some $f \in G$ are *principal ℓ-ideals*. Write Prin-ℓ-Id(G) for the set of all principal ℓ-ideals of G.

(ii) A *prime ℓ-ideal* is an ℓ-ideal $\mathfrak{p} \subseteq G$ such that for all $f, g \in G$, if $0 \leq f \wedge g$ and $f \wedge g \in G$, then $f \in G$ or $g \in G$. Write ℓ-Spec(G) for the set of all prime ℓ-ideals of G, so in particular $G \in \ell$-Spec(G); this slightly strays away from the standard definition of a prime ideal of an ℓ-group, which assumes $\mathfrak{p} \neq G$.

Lemma 2.4.3. (i) *The ℓ-ideals of G are exactly the kernels of ℓ-group homomorphisms $G \longrightarrow H$ onto ℓ-groups H, and prime ℓ-ideals are exactly the kernels of ℓ-group homomorphisms $G \longrightarrow N$ onto o-groups N.*

(ii) *If $f \in G$, then $\ell(f) = \{g \in G \mid \exists n \in \mathbb{N} \text{ such that } |g| \leq n|f|\}$ and $\ell(f) = \ell(|f|)$, where $|f| := f \vee -f$ is the* absolute value of f. *Moreover,*

$$\ell(f) + \ell(g) = \ell(|f| \vee |g|) \quad \textit{and} \quad \ell(f) \cap \ell(g) = \ell(|f| \wedge |g|);$$

in particular, Prin-ℓ-Id$(G) \cup \{G\}$ *is a bounded and distributive lattice under subset inclusion, with join and meet operations given by sum and intersection (respectively), with bottom element $\{0\}$, and with top element G.*

Proof. Item (i) is [BKW77, Corollaire 2.3.7 and Proposition 2.4.3], and the first part of item (ii) is [BKW77, Corollaire 2.2.4]. For the second part of item (ii), note first that $|f| \geq 0$ for all $f \in G$ (see [BKW77, p. 23]), therefore since $\ell(f) = \ell(|f|)$ it suffices to show that $\ell(f)+\ell(g) = \ell(f\vee g)$ and $\ell(f)\cap\ell(g) = \ell(f\wedge g)$ for $f, g \in G$ such that $0 \leq f, g$. If $f, g \in G$ are such that $0 \leq f, g$, then $\ell(f)\cap\ell(g) = \ell(f\wedge g)$ and $\ell(f\vee g) = \ell(\ell(f)\cup\ell(g))$ by [BKW77, Proposition 2.2.10] (or [AF88, Proposition 1.2.3]), and it will be shown that that $\ell(\ell(f) \cup \ell(g)) = \ell(f) + \ell(g)$. Clearly $\ell(f) + \ell(g) \subseteq \ell(\ell(f) \cup \ell(g))$, and if $a \in \ell(\ell(f) \cup \ell(g))$, then $a^+, a^- \in \ell(\ell(f) \cup \ell(g))$, where $a^+ := a \vee 0$ and $a^- := -a \vee 0$; since $a = a^+ - a^-$ (see [BKW77, p. 22]), it suffices to show that $a^+, a^- \in \ell(f) + \ell(g)$. By [BKW77, Proposition 2.2.3], there exist $h_1 \in \ell(f)$ and $h_2 \in \ell(g)$ such that $|a^+| \leq |h_1| + |h_2|$, therefore it follows by the Riesz decomposition theorem [Dar95, Theorem 3.11] and by convexity of $\ell(f)$ and $\ell(g)$ that $|a^+| \in \ell(f) + \ell(g)$. Since $|a^+| = |a \vee 0| = (a\vee 0)\vee -(a\vee 0) = (a\vee 0\vee -a)\wedge(a\vee 0) = a\vee 0 = a^+$, it follows that $a^+ \in \ell(f)+\ell(g)$, and a similar argument shows that $a^- \in \ell(f) + \ell(g)$. □

Remark 2.4.4. An element $0 < u \in G$ is a *strong order unit* if $\ell(u) = G$; so if G has a strong order unit then Prin-ℓ-Id(G) is a bounded and distributive lattice by Lemma 2.4.3 (ii). For example, the constant function with value 1 is a strong order unit in the ℓ-group $C^*(X)$ of bounded continuous real-valued functions on a topological space X. On the other hand, the ℓ-group of all functions $\mathbb{N} \longrightarrow \mathbb{R}$ has no strong order unit.

Theorem 2.4.5 (Weinberg's theorem)**.** *Let $H \subseteq G$ be a subgroup. The sublattice of G generated by H is a subgroup of G; in particular, if $S \subseteq G$ is any non-empty subset, then the sub-ℓ-group of G generated by S consists of the elements of G of the form*

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} f_{ij}, \tag{2.5}$$

where each f_{ij} is in the subgroup of G generated by S.

Proof. See Theorem 6.7 in [Dar95]; given $a, b \in G$ of the form (2.5), the proof actually shows how to construct $c, d \in G$ of the form (2.5) such that $a + b = c$ and $-a = d$. □

2.4.2 The ℓ-spectrum of an ℓ-group

Much in the same way that the set of prime ideals of a ring is equipped with a spectral topology (see Section 2.2), the set ℓ-Spec(G) of all prime ℓ-ideals of an ℓ-group G also carries a spectral topology, see Definition 2.4.8 and Theorem 2.4.9. Moreover, in analogy to the case of real closed rings, ℓ-Spec(G) serves as a set on which G can be represented as an ℓ-group of functions, and the basis of closed sets for the spectral topology on ℓ-Spec(G) can be recovered using the principal ℓ-ideals of G, see Corollary 2.4.7 and Theorem 2.4.9 (IV), respectively.

Throughout this subsection, G is an ℓ-group.

Proposition 2.4.6 (Abstract Nullstellensatz for ℓ-groups)**.** *If $S \subseteq G$ is any non-empty set, then*

$$\ell(S) = \bigcap \{\mathfrak{p} \in \ell\text{-Spec}(G) \mid S \subseteq \mathfrak{p}\}.$$

Proof. Since the ℓ-group G is abelian, it is *representable*, see the beginning of [AF88, Chapter 4] and [AF88, Corollary 4.1.2], as well as [BKW77, Section 4.2] together with [BKW77, Proposition 4.2.9] and [BKW77, pp. 26, 1.6.1]. In particular, every regular ℓ-ideal (see [BKW77, Definition 4.2.3]) is prime by [BKW77, Theoreme 4.2.5], therefore

$$\begin{aligned} \ell(S) &\overset{(*)}{=} \bigcap \{I \subseteq G \mid I \text{ is a regular } \ell\text{-ideal and } S \subseteq I\} \\ &\supseteq \bigcap \{I \subseteq G \mid I \text{ is a prime } \ell\text{-ideal and } S \subseteq I\} \\ &\supseteq \bigcap \{I \subseteq G \mid I \text{ is an } \ell\text{-ideal and } S \subseteq I\} = \ell(S), \end{aligned}$$

where $(*)$ holds by [BKW77, pp. 72, 4.2.4]. □

Corollary 2.4.7. *Every ℓ-group G is isomorphic to a sub-ℓ-group of the ℓ-group of all functions $X \longrightarrow N$ for some non-empty set X and some o-group N; moreover, one may take $X = \ell\text{-Spec}(G)$, or $X = \ell\text{-Spec}^*(G) := \{\mathfrak{p} \in \ell\text{-Spec}(G) \mid \mathfrak{p} \neq G\}$, or $X := \ell\text{-Spec}^{\min}(G) := \{\mathfrak{p} \in \ell\text{-Spec}(G) \mid \mathfrak{p}$ is minimal in the poset $(\ell\text{-Spec}(G), \subseteq)\}$.*

Proof. The proof is analogous to that of Lemma 2.3.3. Note first that

$$(0) = \ell(0) \overset{(1)}{=} \bigcap_{\mathfrak{p} \in \ell\text{-Spec}(G)} \mathfrak{p} \overset{(2)}{=} \bigcap_{\mathfrak{p} \in \ell\text{-Spec}^*(G)} \mathfrak{p} \overset{(3)}{=} \bigcap_{\mathfrak{p} \in \ell\text{-Spec}^{\min}(G)} \mathfrak{p},$$

where (1) follows by Proposition 2.4.6, and (2) and (3) are clear by definition of $\ell\text{-Spec}^*(G)$ and $\ell\text{-Spec}^{\min}(G)$, respectively. Let now X be any of the sets $\ell\text{-Spec}(G)$, $\ell\text{-Spec}^*(G)$, or $\ell\text{-Spec}^{\min}(G)$; by the above and Lemma 2.4.3 (i), the canonical map

$$G \longrightarrow \prod_{\mathfrak{p} \in X} G/\mathfrak{p}$$

$$f \longmapsto (f/\mathfrak{p})_{\mathfrak{p} \in X}$$

is an ℓ-group embedding. For each $\mathfrak{p} \in X$, the total order of the o-group $G/\mathfrak{p}$ extends canonically to a total order on its divisible hull $\text{d.h.}(G/\mathfrak{p})$ in such a way that $\text{d.h.}(G/\mathfrak{p})$ becomes an o-group and $G/\mathfrak{p} \subseteq \text{d.h.}(G/\mathfrak{p})$ is an o-group embedding, see [Mar02, Lemma 3.1.16]. Since the $\{+, -, 0, \leq\}$-theory of divisible o-groups is complete and has quantifier elimination (see [Mar02, Corollary 3.1.17]), this theory has the joint embedding property by [CK90, Proposition 3.5.11], therefore there exists a divisible o-group N such that $\text{d.h.}(G/\mathfrak{p}) \subseteq N$ for all $\mathfrak{p} \in X$, therefore the composite map

$$G \longrightarrow \prod_{\mathfrak{p} \in X} G/\mathfrak{p} \overset{\subseteq}{\longrightarrow} \prod_{\mathfrak{p} \in X} \text{d.h.}(G/\mathfrak{p}) \overset{\subseteq}{\longrightarrow} \prod_{\mathfrak{p} \in X} N = N^X$$

is an ℓ-group isomorphism onto its image, as required. □

Definition 2.4.8. Define the *ℓ-spectrum of G* to be the set $\ell\text{-Spec}(G)$ of all prime ℓ-ideals of G equipped with the topology given by taking the sets

$$D(f) := \{\mathfrak{p} \in \ell\text{-Spec}(G) \mid f \notin \mathfrak{p}\}$$

as a subbasis of open sets with $f \in G$; write also $V(f) := \{\mathfrak{p} \in \ell\text{-Spec}(G) \mid f \in \mathfrak{p}\}$.

Theorem 2.4.9. *The topological space $\ell\text{-Spec}(G)$ is a spectral space and the assignment $G \mapsto \ell\text{-Spec}(G)$ yields a contravariant functor $\ell\text{-Spec} : \ell\text{-}\mathbf{Gp} \longrightarrow \mathbf{Spec}$. Moreover:*

(I) $\mathring{\mathcal{K}}(\ell\text{-Spec}(G)) = \{D(f) \mid f \in G\} \cup \{\ell\text{-Spec}(G)\}$ *and* $\overline{\mathcal{K}}(\ell\text{-Spec}(G)) = \{V(f) \mid f \in G\} \cup \{\varnothing\}$.

(II) $\{G\}$ *is a closed point in* ℓ-Spec(G)*; in particular, the set* $\ell\text{-Spec}^*(G) := \{\mathfrak{p} \in \ell\text{-Spec}(G) \mid \mathfrak{p} \neq G\}$ *of proper prime* ℓ*-ideals of* G *is an open subspace of* ℓ-Spec(G).

(III) *The following are equivalent:*

(i) G *has a strong order unit* (*see* Remark 2.4.4).

(ii) *There exists* $u \in G$ *such that* $D(u) = \ell\text{-Spec}^*(G)$.

(iii) *There exists* $u \in G$ *such that* $V(u) = \{G\}$.

(iv) *The space* $\ell\text{-Spec}^*(G)$ *is quasi-compact.*

(v) $\ell\text{-Spec}^*(G)$ *is proconstructible in* ℓ-Spec(G).

(IV) *The map*

$$\begin{aligned}\overline{\mathcal{K}}(\ell\text{-Spec}(G)) &\longrightarrow \text{Prin-}\ell\text{-Id}(G) \cup \{G\}\\ V(f) &\longmapsto \ell(f)\\ \varnothing &\longmapsto G\end{aligned}$$

is an anti-isomorphism of bounded distributive lattices.

Proof. In [Sch13, Section 2] it is shown that $\{V(f) \mid f \in G\}$ is a subbasis of open sets for a spectral topology on ℓ-Spec(G); and that $\{V(f) \mid f \in G\} \cup \{\varnothing\}$ is the set of quasi-compact open[12] subsets for this topology on ℓ-Spec(G), therefore the first statement of the theorem and item (I) follow from Proposition 2.2.3 (i) and [Sch13, Proposition 2.10]; moreover, $\{G\} = \bigcap_{f\in G} V(f)$, from which (II) follows.

For (III), note that if $u \in G$, then

$$\ell(u) = G \overset{(*)}{\iff} u \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in \ell\text{-Spec}^*(G) \iff D(u) = \ell\text{-Spec}^*(G),$$

[12]Since $G \in \ell$-Spec(G) by definition, it follows that there is no $f \in G$ such that $V(f) = \varnothing$, as otherwise $G \notin V(f)$, that is, $f \notin G$. In particular, some statements in [Sch13] need slight modifications to hold true: for example, [Sch13, Corollary 2.5] holds for non-empty quasi-compact open subsets instead of for all quasi-compact opens.

where $(*)$ follows by Proposition 2.4.6, and this shows the equivalence of (i), (ii), and (iii). The equivalence of (ii) and (iv) is immediate from (I) and from the fact that ℓ-Spec$^*(G)$ is open in ℓ-Spec(G), and the equivalence of (iv) and (v) is obvious.

Finally, since $\ell(f) = \ell(|f|)$ by Lemma 2.4.3 (ii) and $V(f) = V(|f|)$ by [Sch13, Lemma 2.4. (a)] for all $f \in G$, the map in (IV) is bijective; moreover, $\ell(f) = \bigcap_{\mathfrak{p} \in V(f)} \mathfrak{p}$ by Proposition 2.4.6, from which it follows that $V(f) \subseteq V(g)$ if and only if $\ell(g) \subseteq \ell(f)$. $\square$

Remark 2.4.10. The space ℓ-Spec$^*(G)$ of proper prime ℓ-ideals of G is a *generalized spectral space*, that is, it is a T_0 sober space with a basis of quasi-compact opens which is stable under non-empty finite intersections, see for example [CGL99]. Slightly abusing notation, write $\mathring{\mathcal{K}}(\ell\text{-Spec}^*(G)) := \{D(f) \mid f \in G\}$ and $\overline{\mathcal{K}}(\ell\text{-Spec}^*(G)) := \{V(f) \mid f \in G\}$; it follows by Theorem 2.4.9 (I) that $\{D(f) \mid f \in G\}$ is a basis of quasi-compact opens for ℓ-Spec$^*(G)$ which is closed under non-empty finite intersections.

2.4.3 The Shen-Weispfenning theorem

Let G be an ℓ-group. By Corollary 2.4.7 one may regard G as an ℓ-group of functions $X \longrightarrow N$, where X is a non-empty set and N is an o-group. To any such ℓ-group of functions one can associate its *lattice of zero sets* $L_{G,X} := \{\{f \geq 0\} \mid f \in G\}$, where $\{f \geq 0\} := \{x \in X \mid f(x) \geq 0\}$; that each such $\{f \geq 0\}$ is indeed a zero set and $L_{G,X}$ is lattice follows easily from the lattice structure on the ℓ-group G, see Remark 2.4.14. In particular, the ℓ-group of functions G and the lattice of zero sets $L_{G,X}$ can be assembled into a two-sorted model-theoretic structure $(G, L_{G,X})$ connecting the sort of G (called the *home sort*) and the sort of $L_{G,X}$ (called the *space sort*) via the function $G \longrightarrow\!\!\!\!\!\rightarrow L_{G,X}$ given by $f \mapsto \{f \geq 0\}$. Such two-sorted structures are called for brevity *standard structures*, see Definition 2.4.15.

Loosely speaking, the Shen-Weispfenning theorem in [SW87a] (see also [SW87b]) states that if the ℓ-group G of functions $X \longrightarrow N$ is divisible and it satisfies the *patching condition*, then every first order property of the two-sorted structure $(G, L_{G,X})$ can be reduced to a first-order property of the lattice of zero sets $L_{G,X}$; in particular, every first-order property of G can be reduced to a first-order property of $L_{G,X}$. More precisely, the Shen-Weispfenning theorem states that the theory of all standard structures $(G, L_{G,X})$, where G is a divisible ℓ-group of functions $X \longrightarrow N$ which is closed under patching, eliminates quantifiers relative to the space sort, see Theorem 2.4.29.

The patching condition on an ℓ-group G of functions $X \longrightarrow N$ (see Definition 2.4.19) can be seen as stating that a particular kind of formula in the two-sorted structure $(G, L_{G,X})$ is equivalent to a quantifier-free formula. This patching condition is not a property of the ℓ-group G, but rather it is a property of the functional representation $G \subseteq N^X$ of G. In particular, Proposition 2.4.23 shows that if G is an arbitrary ℓ-group, then G is closed under patching when regarded as an ℓ-group of functions on $\ell\text{-Spec}^*(G)$ (see Corollary 2.4.7), therefore the Shen-Weispfenning theorem can be applied to arbitrary divisible ℓ-groups.

This section begins by setting-up the required two-sorted framework following the template given in Subsection 2.1.1, see Definition 2.4.15. This is followed by proving a normal form (♥) for formulas without home quantifiers in standard structures $(G, L_{G,X})$, see Lemma 2.4.17, and then showing that eliminating a home quantifier in formulas of a particular form (♣) reduces to eliminating home quantifiers in formulas of very simple form, see Lemma 2.4.18. After that, the patching condition is introduced in Definition 2.4.19 and both examples and non-examples of ℓ-groups satisfying the patching condition are given. Next, Proposition 2.4.25 shows how the patching condition is used to eliminate home quantifiers in the simple formulas of Lemma 2.4.18, and this should be regarded as the main elimination step in the Shen-Weispfenning theorem. Theorem 2.4.29 (the Shen-Weispfenning theorem) then ties everything together by showing that every formula in divisible standard structures which are closed under patching is equivalent to one of the form (♥). The section concludes with applications of Theorem 2.4.29 to decidability; in particular, Proposition 2.4.34 and Corollary 2.4.35 show how the Shen-Weispfenning theorem applies to the additive ℓ-group reduct of a real closed ring when regarded as an ℓ-group of functions on $\mathrm{Spec}(A)$ (see Lemma 2.3.3).

Definition 2.4.11. Set $\mathscr{L}^{\text{gp}} := \{+, -, 0\}$ to be the language of groups and $\mathscr{L}^{\ell\text{-gp}} := \{+, -, 0, \vee, \wedge, \leq\}$ to be the language of ℓ-groups.

Every ℓ-group can be regarded as an $\mathscr{L}^{\ell\text{-gp}}$-structure in the canonical way. Clearly the class of ℓ-groups is elementary in the language $\mathscr{L}^{\ell\text{-gp}}$.

Corollary 2.4.12. *Every $\mathscr{L}^{\ell\text{-gp}}$-term $t(\overline{x})$ is effectively equivalent modulo the $\mathscr{L}^{\ell\text{-gp}}$-theory of ℓ-groups (see* Definition 2.1.14*) to a term of the form $\bigvee_{i=1}^{m}\bigwedge_{j=1}^{n} t_{ij}(\overline{x})$, where each $t_{ij}(\overline{x})$ is an $\mathscr{L}^{\text{gp}}$-term.*

Proof. Immediate from Theorem 2.4.5 and its proof. □

Definition 2.4.13. Let G be an ℓ-group of functions $X \longrightarrow N$, where X is a non-empty set and N is an o-group. Define $\{f = 0\} := \{x \in X \mid f(x) = 0\}$ to be the *zero-set of f*, and define $\{f \geq 0\}$ and $\{f > 0\}$ analogously.

Remark 2.4.14. Let X be a non-empty set and N be an o-group. For all functions $f, g, h : X \longrightarrow N$ the following identities hold:

$$\{f \vee g \geq h\} = \{f \geq h\} \cup \{g \geq h\}, \;\; \{f \wedge g \geq h\} = \{f \geq h\} \cap \{g \geq h\},$$
$$\{f \vee g \leq h\} = \{f \leq h\} \cap \{g \leq h\}, \text{ and } \{f \wedge g \leq h\} = \{f \leq h\} \cup \{g \leq h\}.$$

Moreover, $\{f = 0\} = \{f \geq 0\} \cap \{-f \geq 0\} = \{f \wedge -f \geq 0\}$ and $\{f \geq 0\} = \{f \wedge 0 = 0\}$.

Definition 2.4.15. (I) Let $\mathscr{L}^{\text{st.str.}}$ be the 2-sorted *language of standard structures*:

- (i) $\Pi \,\dot\cup\, \Sigma$ is a partition of the sorts of $\mathscr{L}^{\text{st.str.}}$, where $\Pi := \{S_{\text{home}}\}$ and $\Sigma := \{S_{\text{space}}\}$ (S_{home} is the *home sort* and S_{space} is the *space sort*).
- (ii) $\mathscr{L}^{\text{st.str.}}_{\restriction \Pi} = \mathscr{L}^{\text{st.str.}}_{\restriction \{S_{\text{home}}\}} := \mathscr{L}^{\ell\text{-gp}}$, and $\mathscr{L}^{\text{st.str.}}_{\restriction \Sigma} = \mathscr{L}^{\text{st.str.}}_{\restriction \{S_{\text{space}}\}} := \mathscr{L}^{\text{lat}}(\top) := \{\sqcup, \sqcap, \sqsubseteq, \top\}$; and
- (iii) $\mathscr{L}^{\text{st.str.}} \setminus (\mathscr{L}_{\restriction \Pi} \,\dot\cup\, \mathscr{L}_{\restriction \Sigma}) := \{\{(-) \geq 0\}\}$, where $\{(-) \geq 0\}$ is a unary function symbol of sort $(S_{\text{home}}, S_{\text{space}})$.

(II) A *standard structure* is an $\mathscr{L}^{\text{st.str.}}$-structure such that:

- (i) the home sort is interpreted as an ℓ-group G of functions $X \longrightarrow N$ (where X is a set and N is an o-group);
- (ii) the space sort is interpreted as the lattice $L_{G,X} := \{\{f \geq 0\} \mid f \in G\}$ of zero sets of $G \subseteq N^X$ (cf. Remark 2.4.14); and
- (iii) the unary function symbol $\{(-) \geq 0\}$ is interpreted as the map $G \longrightarrow\!\!\!\!\rightarrow L_{G,X}$ given by $f \mapsto \{f \geq 0\}$.

(III) The $\mathscr{L}^{\text{st.str.}}$-*theory of standard structures* is the common $\mathscr{L}^{\text{st.str.}}$-theory $T^{\text{st.str.}}$ of all standard structures, i.e.,

$$T^{\text{st.str.}} := \bigcap\{\mathrm{Th}_{\mathscr{L}^{\text{st.str.}}}(\mathscr{G}) \mid \mathscr{G} \text{ is a standard structure}\},$$

where $\mathrm{Th}_{\mathscr{L}^{\text{st.str.}}}(\mathscr{G})$ is the $\mathscr{L}^{\text{st.str.}}$-theory of the $\mathscr{L}^{\text{st.str.}}$-structure $\mathscr{G}$.

Remark 2.4.16. It follows from the functional representation of ℓ-groups given in Corollary 2.4.7 that every ℓ-group has at least one expansion to a standard structure, namely, after identifying G with an ℓ-subgroup of functions $\ell\text{-Spec}^*(G) \longrightarrow N$ for some o-group N, the set $\overline{\mathcal{K}}(\ell\text{-Spec}^*(G))$ is exactly the lattice of zero-sets of G (see Remark 2.4.10), hence the pair $(G, \overline{\mathcal{K}}(\ell\text{-Spec}^*(G)))$ is a standard structure.

Lemma 2.4.17. *Every $\mathscr{L}^{\text{st.str.}}$-formula $\varphi(\overline{z}, \overline{\zeta})$ without home quantifiers is equivalent modulo $T^{\text{st.str.}}$ to an $\mathscr{L}^{\text{st.str.}}$-formula of the form*

$$\exists \xi_1 \dots \xi_m \left[\sigma(\overline{\zeta}, \overline{\xi}) \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{i=1}^{m} \xi_i = \{s_i(\overline{z}) \geq 0\} \right] \tag{♥}$$

where

(i) *$\xi_1, \dots, \xi_m$ are space variables,*

(ii) *$\sigma(\overline{\zeta}, \overline{\xi})$ is a space formula, and*

(iii) *$s_1(\overline{z}), \dots, s_m(\overline{z})$ are $\mathscr{L}^{\text{gr}}$-terms.*

Moreover, if $\varphi(\overline{z}, \overline{\zeta})$ is an existential formula, then the resulting equivalent formula in (♥) is also existential.

Proof. The set of sorts $\Sigma = \{S_{\text{space}}\}$ is closed in $\mathscr{L}^{\text{st.str.}}$ (Definition 2.1.3); moreover, if $s(\overline{z})$ is an $\mathscr{L}^{\text{st.str.}}_{|\text{II}}$-term, then the atomic $\mathscr{L}^{\text{st.str.}}_{|\text{II}}$-formulas $s(\overline{z}) \geq 0$ and $s(\overline{z}) = 0$ are equivalent modulo $T^{\text{st.str.}}$ to $\{s(\overline{z}) \geq 0\} = \top$ and $\{s(\overline{z}) = 0\} = \top$, respectively. Therefore, by Lemma 2.1.8, $\varphi(\overline{z}, \overline{\zeta})$ is equivalent to a formula $\varphi'(\overline{z}, \overline{\zeta})$ of the form (♥) which satisfies items (i) and (ii) in the statement of the lemma, but where each of the terms $s_i(\overline{z})$ are $\mathscr{L}^{\ell\text{-gp}}$-terms.

In order to obtain from $\varphi'(\overline{z}, \overline{\zeta})$ a formula of the form (♥) satisfying all items (i) - (iii) in the statement of the lemma, assume first for notational simplicity and without loss of generality that $m = 1$ in the conjunct appearing in $\varphi'(\overline{z}, \overline{\zeta})$, so that $\varphi'(\overline{z}, \overline{\zeta})$ is the formula

$$\exists\xi\left[\sigma(\overline{\zeta}, \xi) \curlywedge \xi = \{s(\overline{z}) \geq 0\}\right], \tag{2.6}$$

where ξ and $\sigma(\overline{\zeta}, \xi)$ are as in items (i) and (ii) in the statement of the lemma (respectively), and $s(\overline{z})$ is an $\mathscr{L}^{\ell\text{-gr}}$-term. By Corollary 2.4.12, there exist $m_1, m_2 \in \mathbb{N}$ and $\mathscr{L}^{\text{gr}}$-terms $s_{ij}(\overline{z})$ such that $s(\overline{z})$ is equivalent to $\bigvee_{i=1}^{m_1} \bigwedge_{j=1}^{m_2} s_{ij}(\overline{z})$ modulo $T^{\text{st.str.}}$; in particular, (2.6) is equivalent to

$$\exists\xi\left[\sigma(\overline{\zeta}, \xi) \curlywedge \xi = \bigsqcup_{i=1}^{m_1}\bigsqcap_{j=1}^{m_2}\{s_{ij}(\overline{z}) \geq 0\}\right], \tag{2.7}$$

modulo $T^{\text{st.str.}}$ by Remark 2.4.14. Let ξ_{ij} be new space variables for each $i \in [m_1]$ and $j \in [m_2]$; then (2.7) is equivalent to

$$\exists\xi\exists\overline{\xi_{ij}}\left[\sigma(\overline{\zeta}, \xi) \curlywedge \xi = \bigsqcup_{i=1}^{m_1}\bigsqcap_{j=1}^{m_2}\xi_{ij} \curlywedge \bigcurlywedge_{i=1}^{m_1}\bigcurlywedge_{j=1}^{m_2}\xi_{ij} = \{s_{ij}(\overline{z}) \geq 0\}\right], \tag{2.8}$$

and clearly (2.8) is a formula of the form (♥) satisfying all items (i) - (iii) in the statement of the lemma, as required. The moreover part in the statement of the lemma is clear by construction. □

Lemma 2.4.18. *Consider the $\mathscr{L}^{\text{st.str}}$-formula*

$$\exists x\Bigg(\bigcurlywedge_{i\in I_1}\overbrace{\xi_{1i} \sqsubseteq \{x \geq s_{1i}(\overline{z})\}}^{\psi_i(x,\overline{z},\xi_{1i})} \curlywedge \bigcurlywedge_{i\in I_2}\overbrace{\{x \leq s_{2i}(\overline{z})\} \sqsubseteq \xi_{2i}}^{\psi_i(x,\overline{z},\xi_{2i})} \curlywedge$$
$$\bigcurlywedge_{i\in I_3}\underbrace{\xi_{3i} \sqsubseteq \{x \leq s_{3i}(\overline{z})\}}_{\psi_i(x,\overline{z},\xi_{3i})} \curlywedge \bigcurlywedge_{i\in I_4}\underbrace{\{x \geq s_{4i}(\overline{z})\} \sqsubseteq \xi_{4i}}_{\psi_i(x,\overline{z},\xi_{4i})}\Bigg), \tag{♣}$$

where

(i) *x is a home variable;*

(ii) *I_1, I_2, I_3, and I_4 are disjoint finite index sets;*

(iii) *all ξ_{ki} are space variables; and*

(iv) *all $s_{ki}(\overline{z})$ are $\mathscr{L}^{\text{gr}}$-terms.*

Then (♣) is equivalent modulo $T^{\text{st.str}}$ to the conjunction of all the following formulas:

(a) *For all* $i \in I_1$ *and all* $j \in I_3$ *the formula*

$$\exists x \left(\xi_{1i} \sqsubseteq \{x \geq s_{1i}(\overline{z})\} \mathbin{⩕} \xi_{3j} \sqsubseteq \{x \leq s_{3j}(\overline{z})\}\right).$$

(b) *For all* $i \in I_2$ *and all* $j \in I_3$ *the formula*

$$\exists x \left(\{x \leq s_{2i}(\overline{z})\} \sqsubseteq \xi_{2i} \mathbin{⩕} \xi_{3j} \sqsubseteq \{x \leq s_{3j}(\overline{z})\}\right).$$

(c) *For all* $i \in I_1$ *and all* $j \in I_4$ *the formula*

$$\exists x \left(\xi_{1i} \sqsubseteq \{x \geq s_{1i}(\overline{z})\} \mathbin{⩕} \{x \geq s_{4j}(\overline{z})\} \sqsubseteq \xi_{4j}\right).$$

(d) *For all* $i \in I_2$ *and all* $j \in I_4$ *the formula*

$$\exists x \left(\{x \leq s_{2i}(\overline{z})\} \sqsubseteq \xi_{2i} \mathbin{⩕} \{x \geq s_{4j}(\overline{z})\} \sqsubseteq \xi_{4j}\right).$$

Moreover, suppose that G *is an* ℓ*-group of functions* $X \longrightarrow N$ *and let* $\mathscr{G} := (G, L_{G,X})$ *is its corresponding standard stricture. If* $h_{ij} \in G$ $(i \in I_1 \dot\cup I_2,\ j \in I_3 \dot\cup I_4)$ *witness each of the existential quantifiers of the formulas* (i) - (iv) *for* $\mathscr{G}$*, then* $h := \bigvee_{i \in I_1 \dot\cup I_2} \bigwedge_{i \in I_3 \dot\cup I_4} h_{ij}$ *witnesses the existential quantifier of the formula* (♣) *for* $\mathscr{G}$*.*

Proof. Note first that (♣) is the formula

$$\exists x \left(\mathop{⩕}_{k \in [4]} \mathop{⩕}_{i \in I_k} \psi_i(x, \overline{z}, \xi_{ki}) \right) \tag{2.9}$$

where each ψ_i is defined as in the statement of the lemma. Let now G be an arbitrary ℓ-group of functions $X \longrightarrow N$ (where X is a non-empty set and N is an o-group) and set $\mathscr{G} := (G, L_{G,X})$ be its corresponding standard structure. Pick $\overline{d} \in G^{|\overline{z}|}$ and $C_{ki} \in L_{G,X}$ for all for all $k \in [4]$ and $i \in I_k$; for each $k \in [4]$ and $i \in I_k$, define $\varphi_i(x)$ to be the formula (with parameters) $\psi_i(x, \overline{d}, C_{ki})$, and set $e_{ki} := s_{ki}(\overline{d}) \in G$ for all $k \in [4]$ and all $i \in I_k$.

Claim 1. The following are equivalent:

(I) $\mathscr{G} \models \exists x \left(\mathop{⩕}_{k \in [4]} \mathop{⩕}_{i \in I_k} \varphi_i(x) \right)$.

(II) $\mathscr{G} \models \mathop{⩕}_{i \in I_1 \dot\cup I_2} \exists x_i \left(\varphi_i(x_i) \mathbin{⩕} \mathop{⩕}_{j \in I_3 \dot\cup I_4} \varphi_j(x_i) \right)$.

Proof of Claim 1. Clearly (I) implies (II). Conversely, suppose that (II) holds, let $h_i \in G$ $(i \in I_1 \dot{\cup} I_2)$ witness this, and define $h := \bigvee_{i \in I_1 \dot{\cup} I_2} h_i$. If $i \in I_1$, then

$$\{h \geq e_{1i}\} = \left\{ \bigvee_{i' \in I_1 \dot{\cup} I_2} h_{i'} \geq e_{1i} \right\} \overset{(1)}{=} \bigcup_{i' \in I_1 \dot{\cup} I_2} \{h_{i'} \geq e_{1i}\} \overset{(2)}{\supseteq} C_{1i},$$

where (1) follows from Remark 2.4.14, and (2) follows from (II), therefore $\mathscr{G} \models \varphi_i(h)$; similarly, if $i \in I_2$, then

$$\{h \leq e_{2i}\} = \left\{ \bigvee_{i' \in I_1 \dot{\cup} I_2} h_{i'} \leq e_{2i} \right\} = \bigcap_{i' \in I_1 \dot{\cup} I_2} \{h_{i'} \leq e_{2i}\} \subseteq C_{2i}$$

therefore $\mathscr{G} \models \varphi_i(h)$. Analogously, if $j \in I_3$, then

$$\{h \leq e_{3j}\} = \left\{ \bigvee_{i \in I_1 \dot{\cup} I_2} h_i \leq e_{3j} \right\} = \bigcap_{i \in I_1 \dot{\cup} I_2} \{h_i \leq e_{3j}\} \supseteq C_{3j},$$

hence $\mathscr{G} \models \varphi_j(h)$, and if $j \in I_4$, then

$$\{h \geq e_{4j}\} = \left\{ \bigvee_{i \in I_1 \dot{\cup} I_2} h_i \geq e_{4j} \right\} = \bigcup_{i \in I_1 \dot{\cup} I_2} \{h_i \geq e_{4j}\} \subseteq C_{4j}$$

hence $\mathscr{G} \models \varphi_j(h)$; altogether, $\mathscr{G} \models \varphi_i(h)$ for all $i \in I_1 \dot{\cup} I_2 \dot{\cup} I_3 \dot{\cup} I_4$, from which (I) follows. $\square_{\text{Claim 1}}$

Claim 2. The following are equivalent:

(I) $\mathscr{G} \models \bigwedge\!\!\!\bigwedge_{i \in I_1 \dot{\cup} I_2} \exists x_i \left(\varphi_i(x_i) \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{j \in I_3 \dot{\cup} I_4} \varphi_j(x_i) \right)$.

(II) $\mathscr{G} \models \bigwedge\!\!\!\bigwedge_{i \in I_1 \dot{\cup} I_2} \bigwedge\!\!\!\bigwedge_{j \in I_3 \dot{\cup} I_4} \exists x_{ij} \left(\varphi_i(x_{ij}) \wedge\!\!\!\wedge \varphi_j(x_{ij}) \right)$.

Proof of Claim 2. Clearly (I) implies (II). Conversely, suppose that (II) holds, let $h_{ij} \in G$ $(i \in I_1 \dot{\cup} I_2,\ j \in I_3 \dot{\cup} I_4)$ witness this, and define $h_i := \bigwedge_{j \in I_3 \dot{\cup} I_4} h_{ij}$. Fix $i \in I_1 \dot{\cup} I_2$; if $j \in I_3$, then

$$\{h_i \leq e_{3j}\} = \left\{ \bigwedge_{j' \in I_3 \dot{\cup} I_4} h_{ij'} \leq e_{3j} \right\} = \bigcup_{j' \in I_3 \dot{\cup} I_4} \{h_{ij'} \leq e_{3j}\} \supseteq C_{3j},$$

therefore $\mathscr{G} \models \varphi_j(h_i)$, and if $j \in I_4$, then

$$\{h_i \geq e_{4j}\} = \left\{ \bigwedge_{j' \in I_3 \dot{\cup} I_4} h_{ij'} \geq e_{4j} \right\} = \bigcap_{j' \in I_3 \dot{\cup} I_4} \{h_{ij'} \geq e_{4j}\} \subseteq C_{4j},$$

therefore $\mathscr{G} \models \varphi_j(h_i)$. Analogously, if $i \in I_1$, then

$$\{h_i \geq e_{1i}\} = \left\{ \bigwedge_{j \in I_3 \dot\cup I_4} h_{ij} \geq e_{1i} \right\} = \bigcap_{j \in I_3 \dot\cup I_4} \{h_{ij} \geq e_{1i}\} \supseteq C_{1i},$$

hence $\mathscr{G} \models \varphi_i(h_i)$, and if $i \in I_2$, then

$$\{h_i \leq e_{2i}\} = \left\{ \bigwedge_{j \in I_3 \dot\cup I_4} h_j \leq e_{2i} \right\} = \bigcup_{j \in I_3 \dot\cup I_4} \{h_j \leq e_{2i}\} \subseteq C_{2i},$$

hence $\mathscr{G} \models \varphi_i(h_i)$; altogether $\mathscr{G} \models \varphi_j(h_i)$ for all $j \in I_3 \dot\cup I_4$ and all $i \in I_1 \dot\cup I_2$, and $\mathscr{G} \models \varphi_i(h_i)$ for all $i \in I_1 \dot\cup I_2$, therefore (I) follows. $\square_{\text{Claim 2}}$

The statement in the lemma now follows by combining Claim 1 and Claim 2. $\square$

Definition 2.4.19. An ℓ-group G of functions $X \longrightarrow N$ (X a set and N an o-group) is *closed under patching* if for all $f, g \in G$ and all $C, D \in L_{G,X}$

$$f_{\restriction C\cap D} = g_{\restriction C \cap D} \implies \exists h \in G \text{ such that } f_{\restriction C} = h_{\restriction C} \text{ and } g_{\restriction D} = h_{\restriction D}.$$

Remark 2.4.20. Let G be an ℓ-group of functions $X \longrightarrow N$ (X a set and N an o-group), and set $\mathscr{G} := (G, L_{G,X})$ be its corresponding standard structure. Then G is closed under patching if and only if

$$\mathscr{G} \models \forall xy \forall \xi\zeta(\xi \sqcap \zeta \sqsubseteq \{x = y\} \to \exists z(\xi \sqsubseteq \{x = z\} \mathrel{\wedge\!\!\!\wedge} \zeta \sqsubseteq \{y = z\})),$$

therefore being closed under patching is an elementary property of standard structures.

Example 2.4.21. Let N be an o-minimal expansion of a real closed field and let G be the ℓ-group of continuous definable functions $X \longrightarrow N$ on some definable $X \subseteq N^m$. It is claimed that G is closed under patching. Let $f, g \in G$ and $C, D \in L_{G,X}$ be such that $f_{\restriction C\cap D} = g_{\restriction C\cap D}$; then the function $h_0 : C \cup D \longrightarrow N$ given by

$$h_0(x) := \begin{cases} f(x) & \text{if } x \in C \\ g(x) & \text{if } x \in D \end{cases}$$

is continuous and definable, therefore by the definable Tietze extension theorem (see Corollary [Dri98, Chapter 8, Section 3, Corollary 3.10]) there exists $h \in G$ extending h_0, and such h verifies the patching condition for G. The same argument shows that the ℓ-group of continuous functions $X \longrightarrow \mathbb{R}$ on a normal space X is closed under patching, see [Wil70, pp. 103, 15.8].

What follows next is a relatively simple example of an ℓ-group represented as an ℓ-group of functions on a subset of its ℓ-spectrum of which is not closed under patching.

Example 2.4.22. Let N be a divisible o-group with at least one proper non-trivial convex subgroup $\mathfrak{p}$ (for example, N could be the additive group of a non-trivial real closed valuation ring and $\mathfrak{p}$ could be its maximal ideal), and let $\pi : N \twoheadrightarrow N/\mathfrak{p}$ be the projection map. Define G to be the fibre product of N with itself along π, that is,

$$G := N \times_{N/\mathfrak{p}} N = \{(a, b) \in N \times N \mid \pi(a) = \pi(b)\}.$$

G is an ℓ-subgroup of $N \times N$, and the kernels of the projections of G onto each of its coordinates yield $\mathfrak{q}_1, \mathfrak{q}_2 \in \ell\text{-Spec}(G)$ such that $G/\mathfrak{q}_1 \cong G/\mathfrak{q}_2 \cong N$; it is claimed that G, regarded as an ℓ-group of functions $\{\mathfrak{q}_1, \mathfrak{q}_2\} \longrightarrow N$ is not closed under patching. Note that the lattice of zero sets of G is exactly $\mathcal{P}(\{\mathfrak{q}_1, \mathfrak{q}_2\})$, where each of the elements of $\mathcal{P}(\{\mathfrak{q}_1, \mathfrak{q}_2\})$ can be realized as the zero sets of the elements $(0, 0)$, $(\varepsilon, \varepsilon)$, $(0, \varepsilon)$, $(\varepsilon, 0) \in G$, where $\varepsilon \in \mathfrak{p} \setminus \{0\}$. Pick now any $a \in N \setminus \mathfrak{p}$ and $\varepsilon \in \mathfrak{p} \setminus \{0\}$ and set C to be the zero set of $(\varepsilon, 0)$, D to be the zero set of $(0, \varepsilon)$, $f := (a, a)$, and $g := (\varepsilon, \varepsilon)$. Then $C \cap D = \varnothing$, hence $f_{\restriction C\cap D} = g_{\restriction C\cap D}$, but there is no $h \in G$ such that $f_{\restriction C} = h_{\restriction C}$ and $g_{\restriction D} = h_{\restriction D}$, since the only function $\{\mathfrak{q}_1, \mathfrak{q}_2\} \longrightarrow N$ satisfying these conditions is $h := (\varepsilon, a)$ and clearly $h \notin G$.

Recall from Corollary 2.4.7 that every ℓ-group admits a representation as an ℓ-group of functions on various subsets $X \subseteq \ell\text{-Spec}(G)$. Example 2.4.22 shows that for some choices of X, the functional representation of G on X does not yield an ℓ-group of functions which is closed under patching; the next lemma shows that one obtains the patching condition for the choice $X := \ell\text{-Spec}^*(G)$:

Proposition 2.4.23 (Tressl, [Tre22])**.** *Let G be any ℓ-group. Then G, regarded as an ℓ-group of functions $\ell\text{-Spec}^*(G) \longrightarrow N$ for some o-group N (see* Remark 2.4.7*) is closed under patching; equivalently, the standard structure $(G, \overline{\mathcal{K}}(\ell\text{-Spec}^*(G)))$ is closed under patching (see* Remark 2.4.16*).*

Proof. Let $f, g \in G$ and $C, D \in \overline{\mathcal{K}}(\ell\text{-Spec}^*(G))$ be such that $f_{\restriction C\cap D} = g_{\restriction C\cap D}$. Pick $h_1, h_2 \in G^{\geq 0}$ such that $C = V(h_1)$ and $D = V(h_2)$; then $V(h_1) \cap V(h_2) \subseteq V(f - g)$, and thus $\ell(f-g) \subseteq \ell(h_1)+\ell(h_2)$ by Theorem 2.4.9 (IV). Pick $e_1 \in \ell(h_1)$ and $e_2 \in \ell(h_2)$ such that $f - g = e_1 + e_2$, and define $h := f - e_1 = g + e_2$; then $C = V(h_1) \subseteq V(e_1)$ and

$D = V(h_2) \subseteq V(e_2)$ again by Theorem 2.4.9 (IV), and thus $C \subseteq V(e_1) = V(f - h)$ and $D \subseteq V(e_2) = V(g - h)$, that is, $f_{\restriction C} = h_{\restriction C}$ and $g_{\restriction D} = h_{\restriction D}$, as required. □

Remark 2.4.24. Let G be any ℓ-group regarded as an ℓ-group of functions on $\ell\text{-Spec}^*(G)$ as done in Proposition 2.4.23. Then it is straightforward to check that the following are equivalent:

(i) The standard structure $(G, \overline{\mathcal{K}}(\ell\text{-Spec}^*(G)))$ is closed under patching.

(ii) The *Chinese remainder theorem for principal ℓ-ideals of G* holds, that is, for all $f, g, h_1, h_2 \in G$, if $f/(\ell(h_1) + \ell(h_2)) = g/(\ell(h_1) + \ell(h_2))$, then there exists $h \in G$ such that $f/\ell(h_1) = h/\ell(h_1)$ and $g/\ell(h_2) = h/\ell(h_2)$ (cf. [BKW77, Lemme 10.6.3]).

(iii) The presheaf $\mathcal{F}_G$ of ℓ-groups on $(\ell\text{-Spec}(G))_{\text{inv}}$ defined on $\mathring{\mathcal{K}}((\ell\text{-Spec}(G))_{\text{inv}})$ by $\mathcal{F}_G(V(f)) := G/\ell(f)$ and $\mathcal{F}_G(\varnothing) := G/G = 0$ is a sheaf (cf. [Sch13, Theorem 3.1]).

Proposition 2.4.25. *Let G be an ℓ-group of functions $X \longrightarrow N$ which is closed under patching. Let $\mathscr{G} := (G, L_{G,X})$ be its corresponding standard structure, and let also $f, g \in G$ and $C, D \in L_{G,X}$.*

(I) *The following are equivalent:*

(i) $\mathscr{G} \models \exists x\,(C \sqsubseteq \{x \geq f\} \mathbin{⩕} D \sqsubseteq \{x \leq g\})$.

(ii) $\mathscr{G} \models C \sqcap D \sqsubseteq \{f \leq g\}$.

(II) *The following are equivalent:*

(i) $\mathscr{G} \models \exists x\,(\{x \leq f\} \sqsubseteq C \mathbin{⩕} D \sqsubseteq \{x \leq g\})$.

(ii) $\mathscr{G} \models D \sqcap \{g \leq f\} \sqsubseteq C \mathbin{⩕} \exists \xi(\xi \sqcup C = \top \mathbin{⩕} \xi \sqcap D \sqsubseteq \{f \leq g\})$.

(III) *The following are equivalent:*

(i) $\mathscr{G} \models \exists x\,(C \sqsubseteq \{x \geq f\} \mathbin{⩕} \{x \geq g\} \sqsubseteq D)$.

(ii) $\mathscr{G} \models C \sqcap \{g \leq f\} \sqsubseteq D \mathbin{⩕} \exists \zeta(\zeta \sqcup D = \top \mathbin{⩕} C \sqcap \zeta \sqsubseteq \{f \leq g\})$.

(IV) *Suppose further that G is divisible. The following are equivalent:*

(i) $\mathscr{G} \models \exists x\,(\{x \leq f\} \sqsubseteq C \curlywedge \{x \geq g\} \sqsubseteq D)$.

(ii) $\mathscr{G} \models \exists\xi\zeta(\xi \sqcup C = \zeta \sqcup D = \top \curlywedge \xi \sqcap \{g \leq f\} \sqsubseteq D \curlywedge \zeta \sqcap \{g \leq f\} \sqsubseteq C \curlywedge \xi \sqcap \zeta \sqsubseteq \{f \leq g\})$.

Proof. For each of the proofs of the items (I) - (IV) below, let $h \in G$ witness the existential home quantifier $\exists x$ of the formula in corresponding subitem (i).

(I). <u>(i) ⇒ (ii).</u> $C \cap D \subseteq \{h \geq f\} \cap \{h \leq g\} \subseteq \{f \leq g\}$.

<u>(ii) ⇒ (i).</u> $C \cap D \subseteq \{f \leq g\} = \{f \vee g = g\}$, therefore since G is closed under patching there exists $h \in G$ such that $C \subseteq \{f \vee g = h\} \subseteq \{h \geq f\}$ and $D \subseteq \{g = h\} \subseteq \{h \leq g\}$.

(II). <u>(i) ⇒ (ii).</u> $D \cap \{g \leq f\} \subseteq \{h \leq g\} \cap \{g \leq f\} \subseteq \{h \leq f\} \subseteq C$. Define $C' := \{h \geq f\}$; then $X = \{h \geq f\} \cup \{h \leq f\} \subseteq C' \cup C$, hence $X = C' \cup C$, and also $C' \cap D \subseteq \{f \leq h\} \cap \{h \leq g\} \subseteq \{f \leq g\}$, therefore $C' \in L_{G,X}$ witnesses the existential space quantifier $\exists\xi$ in the formula of item (ii).

<u>(ii) ⇒ (i).</u> Let $C' \in L_{G,X}$ witness the existential space quantifier $\exists\xi$ in the formula of item (ii). Pick $g_0 \in G$ such that $g_0 \geq 0$ and $D = \{g_0 = 0\}$ and define $f' := (f \vee g) + g_0$. Then

$$C' \cap D \subseteq \{f \leq g\} \cap \{g_0 = 0\} \subseteq \{f' = g\} \subseteq \{f' \leq g\},$$

therefore by (I) there exists $h \in G$ such that $C' \subseteq \{h \geq f'\}$ and $D \subseteq \{h \leq g\}$. It remains to show that $\{h \leq f\} \subseteq C$; note that

$$\{h \leq f\} \overset{(*)}{=} \{h \leq f\} \cap (C' \cup C) \subseteq (\{h \leq f\} \cap C') \cup C,$$

where $(*)$ follows since $C' \cup C = X$, therefore it suffices to show that $\{h \leq f\} \cap C' \subseteq C$. Indeed:

$$\begin{aligned}\{h \leq f\} \cap C' \subseteq \{h \leq f\} \cap \{h \geq f'\} \subseteq \{f' \leq f\} \subseteq \{g_0 = 0\} \cap \{g \leq f\} \\ = D \cap \{g \leq f\} \subseteq C.\end{aligned}$$

(III). This follows from (II) by considering $\exists x\,(\{x \leq -g\} \sqsubseteq D \curlywedge C \sqsubseteq \{x \leq -f\})$.

(IV). <u>(i) ⇒ (ii).</u> Following similar arguments to the ones in the proof of (II), one can easily check that $C' := \{h \geq f\}$ and $D' := \{h \leq g\}$ witness the space quantifiers $\exists\xi$ and $\exists\zeta$ in the formula of item (ii), respectively.

(ii) ⇒ (i). Let $C' := \{h \geq f\}$ and $D' := \{h \leq g\}$ witness the space quantifiers $\exists\xi$ and $\exists\zeta$ in the formula of item (ii), respectively. Pick $g_0 \in G$ such that $C' \cap D' = \{g_0 = 0\}$, and define $f' := (f \vee \frac{f+g}{2}) + g_0$ and $g' := (g \wedge \frac{f+g}{2}) - g_0$. Then

$$C \cap D \subseteq \{f \leq g\} \cap \{g_0 = 0\} \subseteq \{f' = g_0\} \cap \{g' = g_0\} \subseteq \{f' = g'\} \subseteq \{f' \leq g'\},$$

therefore since G is closed under patching there exists $h \in G$ such that $C \subseteq \{f' = h\}$ and $D \subseteq \{g' = h\}$. Note that

$$\{h \leq f\} = \{h \leq f\} \cap (C' \cup C) \subseteq (\{h \leq f\} \cap C') \cup C$$

and

$$\{h \geq g\} = \{h \geq g\} \cap (D' \cup D) \subseteq (\{h \geq g\} \cap D') \cup D,$$

therefore to show that $\{h \leq f\} \subseteq C$ and $\{h \geq g\} \subseteq D$ it suffices to show that $\{h \leq f\} \cap C' \subseteq C$ and $\{h \geq g\} \cap D' \subseteq D$, respectively. Indeed,

$$\begin{aligned} \{h \leq f\} \cap C' &\subseteq \{f' \leq f\} \cap \{g_0 = 0\} \\ &\subseteq \{g \leq f\} \cap \{g_0 = 0\} \\ &\subseteq \{g \leq f\} \cap C' \cap D' \\ &\subseteq C \end{aligned}$$

and

$$\begin{aligned} \{h \geq f\} \cap D' &\subseteq \{g' \geq g\} \cap \{g_0 = 0\} \\ &\subseteq \{f \geq g\} \cap \{g_0 = 0\} \\ &\subseteq \{f \geq g\} \cap C' \cap D' \\ &\subseteq D, \end{aligned}$$

as required. □

Definition 2.4.26. Define $T^{\text{st.str}}_{\text{div+str}}$ to be the theory $T^{\text{st.str}}$ together with the set of $\mathscr{L}^{\text{st.str}}$-sentences expressing that the group is divisible and that the standard structure is closed under patching (see Remark 2.4.20).

Corollary 2.4.27. *Every $\mathscr{L}^{\text{st.str}}$-formula of the form (♣) is equivalent modulo the theory $T^{\text{st.str}}_{\text{div+patch}}$ to an existential formula without home quantifiers.*

Proof. Combine Lemma 2.4.18 together with Proposition 2.4.25. □

Remark 2.4.28. Since the two-sorted language $\mathscr{L}^{\text{st.str}}$ is finite, it is a recursive language, see Definition 2.1.10, Remark 2.1.11, and the discussion at the end of Subsection 2.1.2. In particular, it follows from their respective proofs that each of the equivalences stated in Lemma 2.4.17 and Lemma 2.4.18 are effective modulo $T^{\text{st.str}}$ in the sense of Definition 2.1.14; similarly, each of the equivalences stated in Proposition 2.4.25 are effective modulo $T^{\text{st.str}}_{\text{div+str}}$, and analogously for Corollary 2.4.27.

Theorem 2.4.29 (Shen-Weispfenning theorem, [SW87a])**.** *Every $\mathscr{L}^{\text{st.str}}$-formula $\varphi(\overline{z},\overline{\zeta})$ is effectively equivalent modulo the $\mathscr{L}^{\text{st.str}}$-theory $T^{\text{st.str}}_{\text{div+patch}}$ of divisible standard structures which are closed under patching to a formula of the form (♥). In particular:*

(i) *The $\mathscr{L}^{\text{st.str}}$-theory $T^{\text{st.str}}_{\text{div+patch}}$ has effective elimination of home quantifiers.*

(ii) *The $\mathscr{L}^{\text{st.str}}$-theory $T^{\text{st.str}}_{\text{div+patch}}$ eliminates quantifiers relative to the space sort.*

(iii) *Every existential $\mathscr{L}^{\text{st.str}}$-formula is effectively equivalent modulo $T^{\text{st.str}}_{\text{div+patch}}$ to an existential formula without home quantifiers.*

Proof. It suffices to show that (i) holds. Indeed, since the set of sorts $\Sigma = \{S_{\text{space}}\}$ is closed in $\mathscr{L}^{\text{st.str}}$ (see Definition 2.1.3), (i) implies (ii) by Lemma 2.1.7, and if (i) holds, then the first statement of the theorem follows from Lemma 2.4.17 and Remark 2.4.28; item (iii) will follow immediately from the constructions in the proof of (i).

To prove (i), it suffices in turn to prove by induction on $n \in \mathbb{N}_0$ that every $\mathscr{L}^{\text{st.str.}}$-formula with n home quantifiers is effectively equivalent modulo $T^{\text{st.str.}}_{\text{div+patch}}$ to a formula without home quantifiers. In what follows, every occurrence of "equivalent" means "effectively equivalent modulo $T^{\text{st.str}}_{\text{div+patch}}$"; see also Remark 2.4.28.

The base case is clear, so assume that the statement holds for some $n \in \mathbb{N}_0$ and let $\varphi(\overline{z},\overline{\zeta})$ be an $\mathscr{L}^{\text{st.str}}$-formula with $n+1$ home quantifiers. Since $\varphi(\overline{z},\overline{\zeta})$ is equivalent to a formula in prenex normal form, it can be assumed that $\varphi(\overline{z},\overline{\zeta})$ is already in prenex normal form; in particular, $\varphi(\overline{z},\overline{\zeta})$ is a formula of the form

$$* * * \underbrace{Qx\ \psi_0}_{\psi},$$

where $* * *$ denotes a (possibly empty) block of space quantifiers, the home quantifier Q is either $\exists$ or $\forall$, and ψ is an $\mathscr{L}^{\text{st.str}}$-formula with exactly n home quantifiers whose free space variables contain $\overline{\zeta}$ and whose free home variables are exactly $\overline{z}$ and x. It now suffices to show that ψ is equivalent to a formula without home quantifiers. By inductive hypothesis, ψ_0 is equivalent to a formula without home quantifiers, therefore by Lemma 2.4.17 ψ is equivalent to

$$Qx\left[\exists\xi_1\ldots\xi_m\left(\sigma \curlywedge \bigcurlywedge_{i=1}^{m}\xi_i = \{s_i(\overline{z},x)\geq 0\}\right)\right], \tag{2.10}$$

where

(i) $\xi_1,\ldots,\xi_m$ are space variables,

(ii) σ is a space formula whose free variables are $\xi_1,\ldots,\xi_m$ together with the free space variables of ψ, and

(iii) $s_1(\overline{z},x),\ldots,s_m(\overline{z},x)$ are $\mathscr{L}^{\text{gr}}$-terms.

Case 1: $Q=\exists$. Since σ is a space formula, there is no occurrence of x in σ, therefore (2.10) is equivalent to

$$\exists\xi_1\ldots\xi_m\left(\sigma \curlywedge \underbrace{\exists x\bigcurlywedge_{i=1}^{m}\xi_i = \{s_i(\overline{z},x)\geq 0\}}_{\delta(\overline{z},\overline{\xi})}\right),$$

and it suffices to show that $\delta(\overline{z},\overline{\xi})$ is equivalent to a formula without home quantifiers. Since each $s_i(\overline{z},x)$ is an $\mathscr{L}^{\text{gr}}$-term, $s_i(\overline{z},x)=t_i(\overline{z})\pm n_ix$ for some $\mathscr{L}^{\text{gr}}$-term $t_i(\overline{z})$ and some $n_i\in\mathbb{N}$, it follows that there exists a unique $m_1\in\mathbb{N}_0$ with $m_1\leq m$ such that $\delta(\overline{z},\overline{\xi})$ is equivalent to

$$\exists x\bigcurlywedge_{i=1}^{m_1}\xi_i=\{n_ix\geq t_i(\overline{z})\}\curlywedge\bigcurlywedge_{i=m_1+1}^{m}\xi_i=\{n_ix\leq t_i(\overline{z})\}. \tag{2.11}$$

Since $\{u_1\geq u_2\}=\{nu_1\geq nu_2\}$ in $T^{\text{st.str}}$ for all home terms u_1,u_2 and all $n\in\mathbb{N}$, it follows that (2.11) is equivalent to

$$\exists x\bigcurlywedge_{i=1}^{m_1}\xi_i=\{nx\geq t_i'(\overline{z})\}\curlywedge\bigcurlywedge_{i=m_1+1}^{m}\xi_i=\{nx\leq t_i'(\overline{z})\}, \tag{2.12}$$

where n is the least common multiple of $n_1,\ldots,n_m$, and $t_i'(\overline{z}):=(n/n_i)\cdot t_i(\overline{z})$. Since the home sort is a divisible group in every model of $T^{\text{st.str}}_{\text{div+patch}}$, (2.12) is equivalent to

$$\exists x\bigcurlywedge_{i=1}^{m_1}\xi_i=\{x\geq t_i'(\overline{z})\}\curlywedge\bigcurlywedge_{i=m_1+1}^{m}\xi_i=\{x\leq t_i'(\overline{z})\}, \tag{2.13}$$

and (2.13) is in turn equivalent to

$$\exists x \Bigg(\bigwedge_{i=1}^{m_1} \xi_i \sqsubseteq \{x \geq t_i'(\overline{z})\} \wedge \bigwedge_{i=m_1+1}^{m} \{x \leq t_i'(\overline{z})\} \sqsubseteq \xi_i \\ \bigwedge_{i=m_1+1}^{m} \xi_i \sqsubseteq \{x \leq t_i'(\overline{z})\} \wedge \bigwedge_{i=1}^{m_1} \{x \geq t_i'(\overline{z})\} \sqsubseteq \xi_i \Bigg). \tag{2.14}$$

Finally, (2.14) is a formula of the form (♣), therefore it is equivalent to an existential formula without home quantifiers by Corollary 2.4.27.

Case 2: $Q = \forall$. In this case (2.10) is equivalent to

$$\neg \; \exists x \; \underbrace{\neg \left[\exists \xi_1 \ldots \xi_m \left(\sigma \wedge \bigwedge_{i=1}^{m} \xi_i = \{s_i(\overline{z}, x) \geq 0\} \right) \right]}_{\psi'}. \tag{2.15}$$

Since ψ' is a formula without home quantifiers, one can proceed as in Case 1 above to eliminate the home quantifier $\exists x \psi'$, form which it follows that (2.15) is equivalent to a formula without home quantifiers. □

The remaining part of this subsection collects some direct consequences of Theorem 2.4.29 on decidability issues; note that Theorem 2.4.29 can also be used to give sufficient conditions for an embedding of standard structures to be existential and elementary.

Corollary 2.4.30. *Let G be a divisible ℓ-group of functions $X \longrightarrow N$ which is closed under patching. Let $\mathscr{G} := (G, L_{G,X})$ be its corresponding standard structure. If the $\mathscr{L}^{\mathrm{lat}}(\top)$-theory of the lattice of zero sets $L_{G,X}$ of G is decidable, then the $\mathscr{L}^{\mathrm{st.str}}$-theory of $\mathscr{G}$ is decidable, and thus so is the $\mathscr{L}^{\ell\text{-gp}}$-theory of G.*

Proof. Let φ be an $\mathscr{L}^{\mathrm{st.str}}$-sentence. Since φ does not have any free variables, it follows by Theorem 2.4.29 that φ is effectively equivalent modulo $\mathscr{G}$ to an $\mathscr{L}^{\mathrm{lat}}(\top)$-sentence σ, therefore $\mathscr{G} \models \varphi$ if and only if $L_{G,X} \models \sigma$, from which the statement follows. □

Corollary 2.4.31. *Let G be any divisible ℓ-group. If the $\mathscr{L}^{\mathrm{lat}}(\top)$-theory of the lattice $\overline{\mathcal{K}}(\ell\text{-Spec}^*(G))$ is decidable, then the $\mathscr{L}^{\ell\text{-gp}}$-theory of G is decidable.*

Proof. Combine Lemma 2.4.16, Proposition 2.4.23, and Corollary 2.4.30. □

The next proposition gives a particular instance in which the lattice of zero sets of an ℓ-group of functions is decidable; this result will be deployed again in Chapter 3.

Proposition 2.4.32. *Let N be an o-minimal expansion of a divisible o-group.*

(i) *Let L_N be the lattice of zero sets of the ℓ-group of continuous definable functions $N \longrightarrow N$. The $\mathscr{L}^{\mathrm{lat}}(S)$-theory of (L_N, S) is decidable for every finite set $S \subseteq L_N$ of constants.*

(ii) *Let $X \subseteq N^m$ be definable and of o-minimal dimension 1. The lattice L_X of zero sets of the ℓ-group of continuous definable functions $X \longrightarrow N$ is isomorphic to a lattice which is parametrically definable in the lattice L_N. In particular, the $\mathscr{L}^{\mathrm{lat}}(S)$-theory of (L_X, S) is decidable for every finite set $S \subseteq L_X$ of constants.*

Proof. A proof analogous to that of Lemma 2.3.34 shows that L_R and L_X are exactly the lattices of closed and parametrically definable subsets of R and X, respectively. In particular, L_R is the lattice of finite unions of intervals $[a, b] \subseteq R \cup \{\pm\infty\}$, where $a, b \in R \cup \{\pm\infty\}$ and $a < b$. Item (i) now follows by [Tre17, Corollary 3.6], and item (ii) follows by [Tre16, 4.1. (vii), items (a) and (c)]. □

Corollary 2.4.33. *Let N be an o-minimal expansion of a real closed field and let $X \subseteq N^m$ be a definable and of o-minimal dimension 1. The $\mathscr{L}^{\ell\text{-gp}}$-theory of the ℓ-group of continuous definable functions $X \longrightarrow N$ is decidable.*

Proof. Combine Example 2.4.21, Proposition 2.4.32, and Corollary 2.4.30. □

Proposition 2.4.34. *Let A be a real closed ring and regard it as an f-ring of functions* $\mathrm{Spec}(A) \longrightarrow R$ *for some real closed field R (see* Lemma 2.3.3*). Then the additive ℓ-group reduct of A is closed under patching.*

Proof. Note first that an element $V \subseteq \mathrm{Spec}(A)$ is a zero set of $A \subseteq R^{\mathrm{Spec}(A)}$ if and only if there exists $f \in A$ such that $V = \{f = 0\} = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid f/\mathfrak{p} = 0\} = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid f \in \mathfrak{p}\} = V(f)$. By Proposition 2.3.4 it follows that the lattice $L_{A,\mathrm{Spec}(A)}$ of zero sets of A is exactly $\overline{\mathcal{K}}(\mathrm{Spec}(A))$. The proof of the statement is now analogous to the proof of Proposition 2.4.23 using Proposition 2.3.4 instead of Theorem 2.4.9 (IV). □

Corollary 2.4.35. *Let A be a real closed ring. If the $\mathscr{L}^{\mathrm{lat}}(\top)$-theory of $\overline{\mathcal{K}}(\mathrm{Spec}(A))$ is decidable, then the $\mathscr{L}^{\ell\text{-gp}}$-theory of the additive ℓ-group reduct of A is decidable.*

Proof. By [Sch89, Chapter I, Corollary 3.6], every real closed ring is an $\mathbb{R}_{\text{alg}}$-algebra, where $\mathbb{R}_{\text{alg}}$ is the field of real algebraic numbers. In particular, the additive ℓ-group reduct of A is a divisible abelian ℓ-group, therefore the statement follows by Proposition 2.4.34 and Corollary 2.4.30. □

Chapter 3

The Lattice-Ordered Module $C_{\textbf{s.a.}}(X)$

Fix the following conventions and notation for this chapter:

(i) Every module is a left module.

(ii) R is a real closed field and $X \subseteq R^m$ is a semi-algebraic curve (Definition 2.3.19).

(iii) $C_{\text{s.a.}}(X) := \{f : X \longrightarrow R \mid f \text{ is continuous and semi-algebraic}\}$.

3.1 Introduction

The set $C_{\text{s.a.}}(X)$ of continuous semi-algebraic functions on a semi-algebraic curve X has the structure of a ring under pointwise addition and multiplication of functions. The ring $C_{\text{s.a.}}(X)$ is real closed (Definition 2.3.1) and its underlying additive group is a divisible ℓ-group (Definition 2.4.1). The semi-algebraic Tietze extension theorem (Theorem 2.3.17) implies that $C_{\text{s.a.}}(X)$ is closed under patching (Definition 2.4.19 and Example 2.4.21), therefore the first-order properties of $C_{\text{s.a.}}(X)$ regarded as an ℓ-group can be effectively reduced to first-order properties of its lattice of zero sets L_X by the Shen-Weispfenning theorem (Subsection 2.4.3). In particular, decidability of the first-order theory of the lattice L_X (Proposition 2.4.32 (ii)) implies that the first-order theory of the ℓ-group $C_{\text{s.a.}}(X)$ is decidable (Corollary 2.4.33).

The goal of this chapter is to adapt the proof technique of the Shen-Weispfenning theorem to show that the first-order properties of $C_{\text{s.a.}}(X)$, regarded this time as a lattice-ordered module over itself in the language of ℓ-groups enriched by all scalar multiplication functions $f \cdot (-)$ for $f \in C_{\text{s.a.}}(X)$, can be reduced to a Boolean combination of first-order properties of its lattice of zero sets L_X and first-order properties of the ring of germs $\mathcal{O}_R$ of continuous semi-algebraic functions $X \longrightarrow R$ at a half-branch of X (Subsection 2.3.2). See Theorem 3.1.8 for the precise formulation of this statement. This reduction of first-order properties is effective under the additional hypothesis that R is a recursive real closed field; this, together with the fact that L_X and $\mathcal{O}_R$ are decidable structures, implies that the first-order theory of the lattice-ordered module $C_{\text{s.a.}}(X)$ is decidable, see Section 3.5. Some further remarks on decidability can be found in Section 3.6, as well as a discussion on where the problems arise in attempting to use the model-theoretic machinery of this chapter to analyze the ring structure on $C_{\text{s.a.}}(X)$.

It will now be explained how the ring of germs $\mathcal{O}_R$ arises naturally in trying to adapt the Shen-Weispfenning theorem to the lattice-ordered module

$$M_X := (C_{\text{s.a.}}(X); +, -, 0, \leq, \vee, \wedge, \{f \cdot (-)\}_{f \in C_{\text{s.a.}}(X)}).$$

Following the construction in Subsection 2.4.3, the first step to adapt the Shen-Weispfenning theorem to M_X is to enrich it with a new sort for the lattice of zero sets L_X of $C_{\text{s.a.}}(X)$ and with the map $M_X \twoheadrightarrow L_X$ given by $f \mapsto \{f \geq 0\}$.

In the resulting two-sorted structure (M_X, L_X) one can express the statement that $f \in C_{\text{s.a.}}(X)$ divides $g \in C_{\text{s.a.}}(X)$ with the formula $\exists y(g = f \cdot y)$. An example in which $(M_X, L_X) \not\models \exists y(g = f \cdot y)$ will be now given in order to see how $\mathcal{O}_R$ appears with this set-up; the role of the lattice L_X in this context will be clarified in the next two paragraphs. Suppose that $X := R$. Let $f \in C_{\text{s.a.}}(X)$ be the absolute value map (so $f(x) := |x|$ for all $x \in X$), and $g \in C_{\text{s.a.}}(X)$ be the identity map (so $g(x) := x$ for all $x \in X$). Then $(M_X, L_X) \not\models \exists y(g = f \cdot y)$ for this choice of X, f, and g, since any function $h : X \longrightarrow R$ satisfying $g = f \cdot h$ must also satisfy that $h_{\restriction[-1,0]} = -1$ and $h_{\restriction[0,1]} = 1$, and thus no such function can be continuous, hence in particular $h \notin C_{\text{s.a.}}(X)$.

On the other hand, note that $(M_X, L_X) \models \exists y([-1, 0] \sqsubseteq \{g = f \cdot y\})$ and also $(M_X, L_X) \models \exists y([0, 1] \sqsubseteq \{g = f \cdot y\})$ with the same choice of X, f and g as above; in other words, f divides g on the curve intervals $[-1, 0]$ and $[0, 1]$ of the half-branches 0^- and 0^+ of $0 \in X$, respectively (see Definition 2.3.29 (ii) for the definition of a curve interval of a half-branch). In particular, even though the divisibility problem $\exists y(g = f \cdot y)$ can be solved on each half-branch 0^- and 0^+ of 0 separately, that is, $\mathcal{O}_R \models \exists Y([g]_{0^-} = [f]_{0^-}Y)$ and $\mathcal{O}_R \models \exists Y([g]_{0^+} = [f]_{0^+}Y)$ (where $[-]_{0^\pm} : C_{\text{s.a.}}(X) \twoheadrightarrow \mathcal{O}_R$ is the germ map at $0^\pm$, see Proposition 2.3.35), these solutions cannot be glued together to a solution locally around the point $0 \in X$.

The example above is an instantiation of the following result which holds for any semi-algebraic curve X and any $f, g \in C_{\text{s.a.}}(X)$ (a slightly stronger version of this next statement is proved in Lemma 3.3.1): there exists $h \in C_{\text{s.a.}}(X)$ such that $g = f \cdot h$ if and only if $\{f = 0\} \subseteq \{g = 0\}$ and f divides g locally around the boundary points of $\{f = 0\}$. The statement that f divides g locally around the boundary points of $\{f = 0\}$ can be expressed as a first-order statement purely in the ring $\mathcal{O}_R$ by referring to the germs $[f]_\beta$ and $[g]_\beta$ of f and g (respectively) at each of the half-branches β of the finitely many boundary points of $\{f = 0\}$ (the fact that there are finitely many such points follows by o-minimality and from the fact that X is 1-dimensional), see Lemma 3.3.2. In this way, the first-order statement $\exists y(g = f \cdot y)$ in M_X is equivalent to the statement $\{f = 0\} \subseteq \{g = 0\}$ expressible in the lattice of zero sets L_X together with a first-order statement expressible in the ring of germs $\mathcal{O}_R$. This suggests to enrich the two-sorted structure (M_X, L_X) with a sort for the ring of germs $\mathcal{O}_R$ and with the germ maps $[-]_\beta : C_{\text{s.a.}}(X) \twoheadrightarrow \mathcal{O}_R$ for all half-branches β at all points of X. This is exactly what is described in the next subsection.

3.1.1 The set-up

Definition 3.1.1 (Page 128 in [Ste10])**.** Let A be a poring. A *lattice-ordered A-module* (*ℓ-A-module* for short) is an ℓ-group $(M, +, -, 0, \leq)$ such that M is an A-module and

$$(f \geq 0 \text{ and } g \geq 0) \implies f \cdot g \geq 0$$

for all $f \in A$ and $g \in M$. An *f-A-module* is an ℓ-A-module M such that

$$(g_1 \wedge g_1 = 0 \text{ and } f \geq 0) \implies (f \cdot g_1) \wedge g_2 = 0 \tag{3.1}$$

for all $f \in A$ and all $g_1, g_2 \in M$.

In particular, since $C_{\text{s.a.}}(X)$ is a real closed ring, it is also an f-ring, therefore $C_{\text{s.a.}}(X)$ has the structure of an f-$C_{\text{s.a.}}(X)$-module, too.

Definition 3.1.2. Let A be a ring. Define $\mathscr{L}^{A\text{-mod}} := \mathscr{L}^{\text{gp}} \,\dot{\cup}\, \{f \cdot (-) \mid f \in A\}$ to be the language of A-modules and $\mathscr{L}^{\ell\text{-}A\text{-mod}} := \mathscr{L}^{\ell\text{-gp}} \,\dot{\cup}\, \{f \cdot (-) \mid f \in A\}$ to be the language of ℓ-A-modules, where each $f \cdot (-)$ is a unary function symbol.

Definition 3.1.3. Let $\mathscr{L}_X$ be the following 3-sorted language:

(i) $\Pi \,\dot{\cup}\, \Sigma$ is a partition of the sorts of the language $\mathscr{L}_X$, where $\Pi := \{S_{\text{home}}\}$ and $\Sigma := \{S_{\text{space}}, S_{\text{germ}}\}$; S_{home} is the *home sort*, S_{space} is the *space sort*, and S_{germ} is the *germ sort*. Home variables are denoted by $x, y, \ldots$, space variables are denoted by $\xi, \zeta, \ldots$, and germ variables are denoted by $Y, Z, \ldots$.

(ii) $\mathscr{L}_{X\restriction\Pi} = \mathscr{L}_{X\restriction\{S_{\text{home}}\}} := \mathscr{L}^{\ell\text{-}C_{\text{s.a.}}(X)\text{-mod}}$.

(iii) $\mathscr{L}_{X\restriction\Sigma} := \mathscr{L}_{X\restriction\{S_{\text{space}}\}} \,\dot{\cup}\, \mathscr{L}_{X\restriction\{S_{\text{germ}}\}}$, where

$$\mathscr{L}_{X\restriction\{S_{\text{space}}\}} := \mathscr{L}^{\text{lat}}(\top, \bot) \quad \text{and} \quad \mathscr{L}_{X\restriction\{S_{\text{germ}}\}} := \mathscr{L}^{\text{ring}}(\leq, \mathtt{m}),$$

where $\top$ and $\bot$ are two constant symbols, $\leq$ is a binary predicate, and $\mathtt{m}$ is a unary predicate.

(iv) $\mathscr{L}_X \setminus (\mathscr{L}_{X\restriction\Pi} \,\dot{\cup}\, \mathscr{L}_{X\restriction\Sigma}) := \{\{(-) \geq 0\}\} \,\dot{\cup}\, \{[(-)]_\beta \mid \beta \text{ is a half-branch of } X\}$, where $\{(-) \geq 0\}$ is a unary function symbol of sort $(S_{\text{home}}, S_{\text{space}})$, and $[(-)]_\beta$ is a unary function symbol of sort $(S_{\text{home}}, S_{\text{germ}})$ for every half-branch β of X.

Definition 3.1.4. Let $\mathscr{M}_X$ be the following $\mathscr{L}_X$-structure:

(i) Interpret the home sort by $C_{\text{s.a.}}(X)$ regarded canonically as an $\mathscr{L}_{X\restriction\Pi}$-structure. In particular, each unary function symbol $f \cdot (-) \in \mathscr{L}_{X\restriction\Pi}$ is interpreted as scalar multiplication by $f \in C_{\text{s.a.}}(X)$.

(ii) Interpret the space sort as the lattice L_X of closed semi-algebraic subsets of X and interpret the germ sort as the ring $\mathcal{O}_R$ of germs of continuous semi-algebraic functions at a half-branch, each regarded canonically as a structure in the corresponding language. In particular, $\top^{\mathscr{M}_X} := X$, $\bot^{\mathscr{M}_X} := \varnothing$, $\leq\, \in \mathscr{L}_{X\restriction\{S_{\text{germ}}\}}$ is interpreted as the total order on $\mathcal{O}_R$, and $\mathtt{m}^{\mathscr{M}_X}$ is the unique maximal ideal $\mathfrak{m}$ of $\mathcal{O}_R$.

(iii) Interpret the unary function symbol $\{(-) \geq 0\}$ as the map $C_{\text{s.a.}}(X) \twoheadrightarrow L_X$ given by $f \longmapsto \{f \geq 0\}$, and for each half-branch β of X, interpret the unary function symbol $[(-)]_\beta$ as the germ map $C_{\text{s.a.}}(X) \twoheadrightarrow \mathcal{O}_R$ given by $f \longmapsto [f]_\beta$.

Here is a diagrammatic view of the 3-sorted structure $\mathscr{M}_X$:

$$
\begin{array}{ccc}
 & & (L_X, \sqsubseteq, \sqcup, \sqcap, \top, \bot) \\
 & \nearrow^{\{(-)\,\geq\, 0\}} & \\
(C_{\text{s.a.}}(X); +, -, 0, \leq, \vee, \wedge, \{f \cdot (-)\}_{f \in C_{\text{s.a.}}(X)}) & & \\
 & \searrow_{\{[(-)]_\beta \;\mid\; \beta \text{ a half-branch of } X\}} & \\
 & & (\mathcal{O}_R, +, -, \cdot, 0, 1, \leq, \mathtt{m})
\end{array}
$$

The statement for $\mathscr{M}_X$ analogous to the Shen-Weispfenning theorem would be that $\mathscr{M}_X$ eliminates quantifiers relative to the space and germ sorts; equivalently (see Lemma 2.1.7), that every $\mathscr{L}_X$-formula is equivalent modulo $\mathscr{M}_X$ to an $\mathscr{L}_X$-formula without home quantifiers. Unfortunately, this is not true as stated. The issue is that in order to eliminate home quantifers in $\mathscr{L}_X$-formulas modulo $\mathscr{M}_X$ one cannot avoid using parameters for elements in the space and germs sorts as shown in Proposition 3.1.6 below, and these parameters are not part of the language $\mathscr{L}_X$. Therefore to obtain the desired relative quantifier elimination result one must enrich the language $\mathscr{L}_X$ with constants for the elements in the lattice of zero sets L_X and the elements in the ring of germs $\mathcal{O}_R$, see Subsection 3.1.2.

Example 3.1.5. Let $f \in C_{\text{s.a.}}(X)$. The zero set $\{f = 0\} \in L_X$ is definable in $\mathscr{M}_X$ by the formula $\exists x(\bot = \{x \leq 0\} \wedge\!\!\!\wedge \{f \cdot x = 0\} = \zeta)$, and since $\bot$ is $\varnothing$-definable in L_X, it follows that $\{f = 0\}$ is $\varnothing$-definable in $\mathscr{M}_X$. Similarly, for every half-branch β of X, the germ $[f]_\beta \in \mathcal{O}_R$ is $\varnothing$-definable in $\mathscr{M}_X$ by the formula $\exists x([x]_\beta = 1 \wedge\!\!\!\wedge [f \cdot x]_\beta = Z)$.

Proposition 3.1.6. *There exist $f, g \in C_{\text{s.a.}}(X)$ and a half-branch β of X such that the formulas*

$$\exists x(\bot = \{x \leq 0\} \wedge\!\!\!\wedge \{f \cdot x = 0\} = \zeta) \quad \textit{and} \quad \exists x([x]_\beta = 1 \wedge\!\!\!\wedge [g \cdot x]_\beta = Z) \tag{3.2}$$

are not equivalent modulo $\mathscr{M}_X$ to $\mathscr{L}_X$-formulas without home quantifiers. In particular, the 3-sorted structure $\mathscr{M}_X$ does not eliminate quantifiers relative to the space and germ sorts.

Proof. Assume for contradiction that every formula of the form (3.2) is equivalent modulo $\mathscr{M}_X$ to a formula without home quantifiers.

Claim 1. Every formula of the form (3.2) is equivalent modulo $\mathscr{M}_X$ to an $\mathscr{L}_{X\restriction\Sigma}$-formula; in particular, the sets defined by formulas of the form (3.2) are $\varnothing$-definable in the $\mathscr{L}_{X\restriction\Sigma}$-reduct $\mathscr{M}_{X\restriction\Sigma}$ of $\mathscr{M}_X$.

Proof of Claim 1. Note first that the set of sorts Σ is closed in $\mathscr{L}_X$ (see Definition 2.1.3). Moreover, for any $\mathscr{L}_{X\restriction\Pi}$-term $t(\overline{x})$, the atomic $\mathscr{L}_{X\restriction\Sigma}$-formulas $t(\overline{x}) \geq 0$ and $t(\overline{x}) = 0$ are equivalent modulo $\mathscr{M}_X$ to $\{t(\overline{x}) \geq 0\} = \top$ and $\{t(\overline{x}) = 0\} = \top$, respectively. Since formulas of the form (3.2) do not have free variables from the home sort, the claim now follows by assumption and by Lemma 2.1.8. $\square_{\text{Claim 1}}$

By Claim 1, every set defined by a formula of the form (3.2) is a $\varnothing$-definable set in the $\mathscr{L}_{X\restriction\Sigma}$-structure $\mathscr{M}_{X\restriction\Sigma}$. Together with Example 3.1.5 this implies that every element in the $\mathscr{L}_{X\restriction\Sigma}$-structure $\mathscr{M}_{X\restriction\Sigma}$ is a $\varnothing$-definable constant, and this is not the case; for completeness this will be shown explicitly. By [Mar02, Proposition 1.3.5] it suffices to find $f, g \in C_{\text{s.a.}}(X)$, a half-branch β of X, and an $\mathscr{L}_{X\restriction\Sigma}$-automorphism $\alpha : \mathscr{M}_{X\restriction\Sigma} \longrightarrow \mathscr{M}_{X\restriction\Sigma}$ which does not fix $\{f \geq 0\} \in L_X$ nor $[g]_\beta \in \mathcal{O}_R$.

Pick any point $a \in X$ and let β be a half-branch of X at a. Let C be a curve interval of β (that is, a curve interval of X at a such that $C \in \beta$, see Definition 2.3.29), and let $\sigma : [0,1]_R \longrightarrow C$ be a semi-algebraic homeomorphism such that $\sigma(0) = a$. Set also $b := \sigma(1)$ and assume without loss of generality that $C \cap \mathrm{cl}_X(X \setminus C) = \{a, b\}$ (one can always choose $\varepsilon \in (0,1)$ such that $C' \cap \mathrm{cl}_X(X \setminus C') = \{a, c\}$, where $C' := \sigma([0,\varepsilon]_R)$ and $c := \sigma(\varepsilon)$). Let $F : [0,1]_R \longrightarrow [0,1]_R$ be the semi-algebraic homeomorphism given by $x \mapsto x^2$; then $G := \sigma \circ F \circ \sigma^{-1} : C \longrightarrow C$ is a semi-algebraic homeomorphism whose unique fixed points are a and b. In particular, the map $H : X \longrightarrow X$ given by

$$H(x) := \begin{cases} x & \text{if } x \in \mathrm{cl}_X(X \setminus C) \\ G(x) & \text{if } x \in C \end{cases}$$

is a well-defined semi-algebraic bijection. Moreover, if $D \subseteq X$ is closed, then

$$\begin{aligned} H(D) = H([D \cap C] \cup [D \cap \mathrm{cl}_X(X \setminus C)]) &= H(D \cap C) \cup H(D \cap \mathrm{cl}_X(X \setminus C)) \\ &= G(D \cap C) \cup [D \cap \mathrm{cl}_X(X \setminus C)] \end{aligned}$$

is closed in X, and similarly $H^{-1}(D)$ is also closed, therefore H is a semi-algebraic homeomorphism $X \longrightarrow X$.

Claim 2. Write $L_X := \{\{f \geq 0\} \mid f \in C_{\text{s.a.}}(X)\}$ and $\mathcal{O}_R := \{[f]_\beta \mid f \in C_{\text{s.a.}}(X)\}$. The maps $\alpha_1 : L_X \longrightarrow L_X$ and $\alpha_2 : \mathcal{O}_R \longrightarrow \mathcal{O}_R$ given by

$$\alpha_1(\{f \geq 0\}) := \{f \circ H \geq 0\} \quad \text{and} \quad \alpha_2([f]_\beta) := [f \circ H]_\beta$$

are an $\mathscr{L}^{\text{lat}}(\top, \bot)$-automorphism and a $\mathscr{L}^{\text{ring}}(\leq, \mathtt{m})$-automorphism, respectively.

Proof of Claim 2. Let $f, g \in C_{\text{s.a.}}(X)$. Then

$$\begin{aligned}
\{f \geq 0\} = \{g \geq 0\} &\iff f^{-1}(R^{\geq 0}) = g^{-1}(R^{\geq 0}) \\
&\iff H^{-1}(f^{-1}(R^{\geq 0})) = H^{-1}(g^{-1}(R^{\geq 0})) \\
&\iff (f \circ H)^{-1}(R^{\geq 0}) = (g \circ H)^{-1}(R^{\geq 0}) \\
&\iff \alpha_1(\{f \geq 0\}) = \alpha_1(\{g \geq 0\},
\end{aligned}$$

therefore α_1 is a well-defined injective map. Clearly α_1 is also surjective and it preserves the top and bottom elements of L_X; moreover

$$\begin{aligned}
\alpha_1(\{f \geq 0\} \cup \{g \geq 0\}) = \alpha_1(\{f \vee g \geq 0\}) &= \{(f \vee g) \circ H \geq 0\} \\
&= \{(f \circ H) \vee (g \circ H) \geq 0\} \\
&= \{(f \circ H) \geq 0\} \cup \{(g \circ H) \geq 0\} \\
&= \alpha_1(\{f \geq 0\}) \cup \alpha_1(\{g \geq 0\})
\end{aligned}$$

by Remark 2.4.14, and that α_1 preserves binary meets follows analogously, therefore α_1 is a lattice isomorphism.

To prove that α_2 is an $\mathscr{L}^{\text{ring}}(\leq, \mathtt{m})$-automorphism it suffices to show that it is a ring isomorphism, since the interpretations of $\leq$ and $\mathtt{m}$ in $\mathcal{O}_R$ are $\mathscr{L}^{\text{ring}}$-definable subsets. If $[f]_\beta = [g]_\beta$, then there exists a curve interval D of β such that $f_{\restriction D} = g_{\restriction D}$ by Proposition 2.3.35 (iv), and since C is also a curve interval of β it may be assumed that $D \subseteq C$. By construction of H, it follows that $H^{-1}(D) = G^{-1}(D)$ is also a curve interval of β, therefore $D' := H^{-1}(D)$ is a curve interval of β such that $(f \circ H)_{\restriction D'} = f_{\restriction H(D')} = g_{\restriction H(D')} = (g \circ H)_{\restriction D'}$, therefore $[f \circ H]_\beta = [g \circ H]_\beta$ by Proposition 2.3.35 (iv), showing thus that α_2 is a well-defined map. A similar argument shows that α_2 is injective, and clearly α_2 is a surjective ring homomorphism, therefore this concludes the proof. $\square_{\text{Claim 2}}$

By definition of $\mathscr{L}_{X\restriction\Sigma}$, the pair (α_1, α_2) is an automorphism of $\mathscr{M}_{X\restriction\Sigma}$. Pick $c \in C \setminus \{a, b\}$ and $f \in C_{\text{s.a.}}(X)$ be such that $f(c) = 0$ and $f(x) < 0$ for all $x \in X \setminus \{c\}$; then $H(c) = G(c) \neq c$ by construction of G, hence $\{f \geq 0\} = \{c\} \neq \{H(c)\} = \{f \circ H \geq$

$0\} = \alpha_1(\{f \geq 0\})$ and thus α_1 does not fix $\{f \geq 0\} \in L_X$. Now let $g \in C_{\text{s.a.}}(X)$ be such that g does not take the same value twice on C. Then $[g]_\beta \neq [g \circ H]_\beta$, as otherwise there would exist a curve interval D of β such that $D \subseteq C$ and $g_{\restriction D} = (g \circ H)_{\restriction D}$, hence $g(d) = g(H(d))$ for some $d \in D \setminus \{a\}$, but $H(d) = G(d) \in C$ and $H(d) \neq d$ by construction of G, contradicting the choice of g, therefore α_2 does not fix $[g]_\beta$. □

3.1.2 Statement of the main theorem and proof outline

As explained at the beginning of Section 3.1, a first suitable adaptation of the Shen-Weispfenning machinery to the lattice-ordered module $C_{\text{s.a.}}(X)$ could be potentially carried for the three-sorted $\mathscr{L}_X$-structure $\mathscr{M}_X$ given in Definitions 3.1.3 and 3.1.4. On the other hand, Proposition 3.1.6 and its proof show that the desired relative quantifier elimination statement for $\mathscr{M}_X$ cannot be achieved unless the language $\mathscr{L}_X$ is enriched by constant symbols for elements in the space and germ sorts. The resulting canonical expansion $\mathscr{M}_X^{\text{const}}$ of $\mathscr{M}_X$ to this enriched language $\mathscr{L}_X^{\text{const}}$ eliminates quantifiers relative to the space and germ sorts, and this is the main theorem of this chapter:

Definition 3.1.7. Let $\mathscr{L}_X^{\text{const}}$ be the enrichment of the language $\mathscr{L}_X$ consisting of adding constant symbols for each element in the lattice of zero sets L_X and for each element in the ring of germs $\mathcal{O}_R$. *Space formulas with parameters* are just $\{S_{\text{space}}\}$-sorted $\mathscr{L}_X^{\text{const}}$-formulas, in other words, space formulas in the language $\mathscr{L}_X$ with parameters from L_X, and *germs formulas with parameters* are defined analogously. Define $\mathscr{M}_X^{\text{const}}$ to be the canonical expansion of $\mathscr{M}_X$ to the language $\mathscr{L}_X^{\text{const}}$.

Theorem 3.1.8. *Every $\mathscr{L}_X^{\text{const}}$-formula $\varphi(\overline{z}, \overline{\zeta}, \overline{Z})$ is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula of the form*

$$\exists \xi_1 \ldots \xi_m \exists Y_1 \ldots Y_n \left[\sigma(\overline{\zeta}, \overline{Z}, \overline{\xi}, \overline{Y}) \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{i=1}^{m} \xi_i = \{t_i(\overline{z}) \geq 0\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{i=1}^{n} Y_j = [s_j(\overline{z})]_{\beta_j} \right] \qquad (\spadesuit)$$

where

(i) *$\xi_1, \ldots, \xi_m$ are space variables and $Y_1, \ldots, Y_n$ are germ variables,*

(ii) *$\sigma(\overline{\zeta}, \overline{Z}, \overline{\xi}, \overline{Y})$ is a Boolean combination of space formulas with parameters and germ formulas with parameters,*

(iii) *$t_1(\overline{z}), \ldots, t_m(\overline{z})$ and $s_1(\overline{z}), \ldots, s_n(\overline{z})$ are $\mathscr{L}^{C_{\text{s.a.}}(X)\text{-mod}}$-terms, and*

(iv) $\beta_1, \dots, \beta_n$ *are half-branches of* X.

In particular, every $\mathscr{L}_X^{\text{const}}$*-formula* $\varphi(\overline{z}, \overline{\zeta}, \overline{Z})$ *is equivalent modulo* $\mathscr{M}_X^{\text{const}}$ *to an* $\mathscr{L}_X^{\text{const}}$*-formula without home quantifiers. Equivalently,* $\mathscr{M}_X^{\text{const}}$ *eliminates quantifiers relative to the space and the germ sorts.*

The proof of 3.1.8 mimics the proof of the Shen-Weispfenning theorem using also the formalism of Subsection 2.1.1, and it is developed in the following steps in Sections 3.2, 3.3, and 3.4:

(i) First it is shown in Lemma 3.2.6 that every $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to a formula of the form (♠).

(ii) After that, it is shown in successive steps (Lemmas 3.2.7, 3.2.8, and 3.2.9) that in order to eliminate home quantifiers in arbitrary $\mathscr{L}_X^{\text{const}}$-formulas modulo $\mathscr{M}_X^{\text{const}}$ it suffices to eliminate home quantifiers in formulas of the form ($♣_2$).

(iii) This is followed by the elimination steps, which consist in showing one can eliminate home quantifiers modulo $\mathscr{M}_X^{\text{const}}$ in formulas of a particular form, namely the formulas ($★_1$) and ($★_2$). This is all the content of Section 3.3

(iv) Section 3.4 ties everything together: step (i) above shows that it suffices to prove that every $\mathscr{L}_X^{\text{const}}$-formula is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to a formula without home quantifiers, step (ii) shows that it suffices in turn to eliminate home quantifiers in formulas of the form ($♣_2$), and in Section 3.4 it is shown how to eliminate home quantifiers in formulas of the form ($♣_2$) using the elimination results from step (iii).

The need for the space and germ parameters is also manifested in the simplification steps in item (ii); in particular, germ parameters arise naturally in the proof of Lemma 3.2.8, and space parameters arise naturally in the proof of Lemma 3.2.9.

The elimination of home quantifiers in formulas ($★_1$) and in formulas ($★_2$) are of different flavours. Formulas of the form ($★_1$) essentially deal with divisibility by fixed scalars just as explained at the beginning of Section 3.1. The main idea to eliminate home quantifiers in such formulas is to use the semi-algebraic Tietze extension theorem to glue together local solutions provided by the ring of germs $\mathcal{O}_R$ to divisibility problems on curve intervals of half-branches of X. On the other hand, $\mathscr{L}_X^{\text{const}}$-formulas of the form ($\bigstar_2$) can be regarded as analogues of the $\mathscr{L}^{\text{st.str}}$-formulas in Lemma 2.4.18 from the Shen-Weispfenning construction. The main idea to eliminate home quantifiers in formulas of the form ($\bigstar_2$) is to reduce it first to eliminating home quantifiers in simpler formulas, and then use certain conditions expressible in the space and germ sorts to show that eliminating home quantifiers in such simpler formulas essentially boils down to an application of Lemma 2.4.18 and Proposition 2.4.25.

3.2 Simplification steps

Lemma 3.2.1. *Let A be a poring and M be an f-A-module. Let $f \in A$ and $g_1, g_2 \in M$. If $f \geq 0$, then $f \cdot (g_1 \wedge g_2) = f \cdot g_1 \wedge f \cdot g_2$ and $f \cdot (g_1 \vee g_2) = f \cdot g_1 \vee f \cdot g_2$.*

Proof. If $h \in M$, then

$$g_1 \wedge g_2 = h \iff (g_1 \wedge g_2) - h = 0 \iff (g_1 - h) \wedge (g_2 - h) = 0; \tag{3.3}$$

in particular,

$$[g_1 - (g_1 \wedge g_2)] \wedge [g_2 - (g_1 \wedge g_2)] = 0. \tag{3.4}$$

Suppose that $f \geq 0$; applying (3.1) twice to (3.4) it follows that

$$f \cdot [g_1 - (g_1 \wedge g_2)] \wedge f \cdot [g_2 - (g_1 \wedge g_2)] = 0,$$

therefore

$$[f \cdot g_1 - f \cdot (g_1 \wedge g_2)] \wedge [f \cdot g_2 - f \cdot (g_1 \wedge g_2)] = 0$$

and thus $f \cdot (g_1 \wedge g_2) = f \cdot g_1 \wedge f \cdot g_2$ follows by the equivalence in (3.3). The identity $f \cdot (g_1 \vee g_2) = f \cdot g_1 \vee f \cdot g_2$ follows from $f \cdot (g_1 \wedge g_2) = f \cdot g_1 \wedge f \cdot g_2$ together with $f \cdot (g_1 \vee g_2) = -[f \cdot [(-g_1) \wedge (-g_2)]]$. ☐

Proposition 3.2.2 (cf. Exercise 6 (g) in page 135 of [Ste10])**.** *Let A be an ℓ-ring, M be an f-A-module, and $N \subseteq M$ be a sub-A-module. The sublattice of M generated by N is a sub-A-module of M; in particular, if $S \subseteq M$ is any non-empty subset, then the sub-ℓ-A-module of M generated by S consists of the elements of M of the form*

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} g_{ij},$$

where each g_{ij} is in the sub-A-module of M generated by S.

Proof. Let N' be the sublattice of M generated by N; N' is a subgroup of M by Weinberg's theorem (Theorem 2.4.5), therefore it suffices to show that $f \cdot g \in N'$ for all $f \in A$ and $g \in N'$. Write $g = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} g_{ij}$ with $g_{ij} \in N$ for all $i \in [m]$ and $j \in [n]$; then

$$\begin{aligned} f \cdot g &= (f^+ - f^-) \cdot \left(\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} g_{ij} \right) && \text{since } A \text{ is an } \ell\text{-ring} \\ &= f^+ \cdot \left(\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} g_{ij} \right) - f^- \cdot \left(\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} g_{ij} \right) && \text{since } M \text{ is an } A\text{-module} \\ &= \left(\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} f^+ \cdot g_{ij} \right) - \left(\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} f^- \cdot g_{ij} \right) && \text{by Lemma 3.2.1.} \end{aligned}$$

Since $f^+ \cdot g_{ij}, f^- \cdot g_{ij} \in N$ for all $i \in [m]$ and $j \in [n]$, and N' is a subgroup of M, $f \cdot g \in N'$ follows, as required. □

Remark 3.2.3. If A is an ℓ-ring, then every element in $A^{\geq 0}$ is of the form $f \vee 0$ for some $f \in A$, therefore the class of ℓ-A-modules is elementary in the language $\mathscr{L}^{\ell\text{-}A\text{-mod}}$. Explicitly, an axiomatization for this class consists of the $\mathscr{L}^{\ell\text{-gp}}$-axioms for ℓ-groups together with the $\mathscr{L}^{A\text{-mod}}$-axioms for A-modules and the set of $\mathscr{L}^{\ell\text{-}A\text{-mod}}$-sentences $\{\forall x[x \geq 0 \to (f \vee 0) \cdot x \geq 0] \mid f \in A\}$. Similarly, the class of f-A-modules is also elementary in the language $\mathscr{L}^{\ell\text{-}A\text{-mod}}$.

Corollary 3.2.4. *Let A be an ℓ-ring. Every $\mathscr{L}^{\ell\text{-}A\text{-mod}}$-term $t(\overline{x})$ is equivalent modulo the $\mathscr{L}^{\ell\text{-}A\text{-mod}}$-theory of f-A-modules to a term of the form $\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} t_{ij}(\overline{x})$, where each $t_{ij}(\overline{x})$ is an $\mathscr{L}^{A\text{-mod}}$-term.*

Proof. Immediate from Proposition 3.2.2. □

Lemma 3.2.5. *Every $\mathscr{L}_X$-formula $\varphi(\overline{z}, \overline{\zeta}, \overline{Z})$ without home quantifiers is equivalent modulo $\mathscr{M}_X$ to an $\mathscr{L}_X$-formula of the form*

$$\exists \xi_1 \dots \xi_m \exists Y_1 \dots Y_n \left[\sigma(\overline{\zeta}, \overline{Z}, \overline{\xi}, \overline{Y}) \mathbin{\&} \bigwedge_{i=1}^{m} \xi_i = \{t_i(\overline{z}) \geq 0\} \mathbin{\&} \bigwedge_{i=1}^{n} Y_j = [s_j(\overline{z})]_{\beta_j} \right] \quad (3.5)$$

where

(i) *$\xi_1, \dots, \xi_m$ are space variables and $Y_1, \dots, Y_n$ are germ variables,*

(ii) *$\sigma(\overline{\zeta}, \overline{Z}, \overline{\xi}, \overline{Y})$ is a Boolean combination of space formulas and germ formulas,*

(iii) $t_1(\overline{z}), \ldots, t_m(\overline{z})$ *and* $s_1(\overline{z}), \ldots, s_n(\overline{z})$ *are* $\mathscr{L}^{C_{\text{s.a.}}(X)\text{-mod}}$*-terms, and*

(iv) $\beta_1, \ldots, \beta_n$ *are half-branches of* X*.*

Proof. The set of sorts $\Sigma = \{S_{\text{space}}, S_{\text{germ}}\}$ is closed in $\mathscr{L}_X$ (Definition 2.1.3); moreover, if $t(\overline{z})$ is an $\mathscr{L}_{X\restriction\Pi}$-term, then the atomic $\mathscr{L}_{X\restriction\Pi}$-formulas $t(\overline{z}) \geq 0$ and $t(\overline{z}) = 0$ are equivalent modulo $\mathscr{M}_X$ to $\{t(\overline{z}) \geq 0\} = \top$ and $\{t(\overline{z}) = 0\} = \top$, respectively. Since $\mathscr{L}_{X\restriction\Sigma} \setminus (\mathscr{L}_{X\restriction\{S_{\text{space}}\}} \dot{\cup} \mathscr{L}_{X\restriction\{S_{\text{germ}}\}}) = \varnothing$, it follows from Lemma 2.1.6 (applied to, say, the closed sort $\{S_{\text{germ}}\}$ in the language $\mathscr{L}_{X\restriction\Sigma}$) that $\mathscr{L}_{X\restriction\Sigma}$-formulas are Boolean combinations of $\mathscr{L}_{X\restriction\{S_{\text{space}}\}}$-formulas and $\mathscr{L}_{X\restriction\{S_{\text{germ}}\}}$-formulas, therefore, by Lemma 2.1.8, $\varphi(\overline{z}, \overline{\zeta}, \overline{Z})$ is equivalent to a formula $\varphi'(\overline{z}, \overline{\zeta}, \overline{Z})$ of the form (3.5) which satisfies items (i), (ii), and (iv) in the statement of the lemma, but where each of the terms $t_i(\overline{z})$ and $s_j(\overline{z})$ are $\mathscr{L}^{\ell\text{-}C_{\text{s.a.}}(X)\text{-mod}}$-terms.

In order to obtain from $\varphi'(\overline{z}, \overline{\zeta}, \overline{Z})$ a formula of the form (3.5) satisfying all items (i) - (iv) in the statement of the lemma, assume for notational simplicity that $m = n := 1$ in the conjuncts appearing in $\varphi'(\overline{z}, \overline{\zeta}, \overline{Z})$ (the case for arbitrary m and n is treated analogously), so that $\varphi'(\overline{z}, \overline{\zeta}, \overline{Z})$ is the formula

$$\exists \xi \exists Y \left[\sigma(\overline{\zeta}, \overline{Z}, \xi, Y) \mathbin{\&\!\!\&} \xi = \{t(\overline{z}) \geq 0\} \mathbin{\&\!\!\&} Y = [s(\overline{z})]_\beta \right], \tag{3.6}$$

where ξ, Y, $\sigma(\overline{\zeta}, \overline{Z}, \xi, Y)$, and β are as in items (i), (ii) and (iv) of the statement of the lemma, but $t(\overline{z})$ and $s(\overline{z})$ are $\mathscr{L}^{\ell\text{-}C_{\text{s.a.}}\text{-mod}}$-terms. By Corollary 3.2.4, there exist $m_1, n_1, m_2, n_2 \in \mathbb{N}$ and $\mathscr{L}^{C_{\text{s.a.}}(X)\text{-mod}}$-terms $t_{ij}(\overline{z})$ and $s_{kl}(\overline{z})$ such that $t(\overline{z})$ is equivalent to $\bigvee_{i=1}^{m_1} \bigwedge_{j=1}^{n_1} t_{ij}(\overline{z})$ and $s(\overline{z})$ is equivalent to $\bigvee_{k=1}^{m_2} \bigwedge_{l=1}^{n_2} s_{kl}(\overline{z})$ modulo $\mathscr{M}_X$; in particular, (3.6) is equivalent to

$$\exists \xi \exists Y \left[\sigma(\overline{\zeta}, \overline{Z}, \xi, Y) \mathbin{\&\!\!\&} \xi = \bigsqcup_{i=1}^{m_1} \bigsqcap_{j=1}^{n_1} \{t_{ij}(\overline{z}) \geq 0\} \mathbin{\&\!\!\&} \underbrace{Y = \max_{1 \leq k \leq m_2} \min_{1 \leq l \leq n_2} [s_{kl}(\overline{z})]_\beta}_{(*)} \right], \tag{3.7}$$

modulo $\mathscr{M}_X$ by Remark 2.4.14 and Theorem 2.3.2 (III), where $(*)$ is a shorthand for the obvious $\mathscr{L}^{\text{ring}}(\leq, \mathtt{m})$-formula. Let ξ_{ij} be new space variables for each $i \in [m_1]$ and $j \in [n_1]$, and let Y_{kl} be new germ variables for each $k \in [m_2]$ and $l \in [n_2]$; then (3.7) is equivalent to

$$\exists\xi\exists\overline{\xi_{ij}}\exists Y\exists\overline{Y_{kl}}\Bigg[\sigma(\overline{\zeta},\overline{Z},\xi,Y)\ \curlywedge\ \xi=\bigsqcup_{i=1}^{m_1}\bigsqcap_{j=1}^{n_1}\xi_{ij}\ \curlywedge\ Y=\max_{1\le k\le m_2}\min_{1\le l\le n_2}Y_{kl}\ \curlywedge$$
$$\bigwedge_{i=1}^{m_1}\bigwedge_{j=1}^{n_1}\xi_{ij}=\{t_{ij}(\overline{z})\geq 0\}\ \curlywedge\ \bigwedge_{k=1}^{m_2}\bigwedge_{l=1}^{n_2}Y_{kl}=[s_{kl}(\overline{z})]_\beta\Bigg], \tag{3.8}$$

modulo $\mathscr{M}_X$, and clearly (3.8) is a formula of the form (3.5) satisfying all items (i) - (iv) in the statement of the lemma, as required. □

Lemma 3.2.6. *Every $\mathscr{L}_X^{\text{const}}$-formula $\varphi(\overline{z},\overline{\zeta},\overline{Z})$ without home quantifiers is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula of the form (♠).*

Proof. Analogous to the proof of Lemma 3.2.5. □

Lemma 3.2.7. *Suppose that every $\mathscr{L}_X$-formula of the form*

$$\exists x\left[\bigwedge_{i=1}^{m}\xi_i=\{t_i(\overline{z},x)\geq 0\}\ \curlywedge\ \bigwedge_{i=1}^{n}Y_j=[s_j(\overline{z},x)]_{\beta_j}\right] \tag{♣}$$

is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers, where

(i) *x is a home variable;*

(ii) *$\xi_1,\dots,\xi_m$ are space variables and $Y_1,\dots,Y_n$ are germ variables;*

(iii) *$t_1(\overline{z},x),\dots,t_m(\overline{z},x)$ and $s_1(\overline{z},x),\dots,s_n(\overline{z},x)$ are $\mathscr{L}^{C_{\text{s.a.}}(X)\text{-mod}}$-terms; and*

(iv) *$\beta_1,\dots,\beta_n$ are half-branches of X.*

Then every $\mathscr{L}_X^{\text{const}}$-formula is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers.

Proof. This follows by writing $\mathscr{L}_X^{\text{const}}$-formulas in prenex normal form and then inducting over home quantifiers using Lemma 3.2.6; see the proof of Proposition 2.1.9 or the proof of Theorem 2.4.29. □

Lemma 3.2.8. *Suppose that that every $\mathscr{L}_X$-formula of the form*

$$\exists x\left[\bigwedge_{i=1}^{m}\xi_i=\{t_i(\overline{z},x)\geq 0\}\ \curlywedge\ \bigwedge_{j=1}^{n}Y_j=[x]_{\beta_j}\right] \tag{♣$_1$}$$

is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers, where

(i) *x is a home variable;*

(ii) *$\xi_1, \dots, \xi_m$ are space variables and $Y_1, \dots, Y_n$ are germ variables;*

(iii) *$t_1(\overline{z}, x), \dots, t_m(\overline{z}, x)$ are $\mathscr{L}^{C_{\text{s.a.}}(X)\text{-mod}}$-terms; and*

(iv) *$\beta_1, \dots, \beta_n$ are half-branches of X.*

Then every $\mathscr{L}_X^{\text{const}}$-formula is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers.

Proof. Let $\varphi(\overline{z}, \overline{\xi}, \overline{Y})$ be an $\mathscr{L}_X$-formula of the form (✽); by Lemma 3.2.7 it suffices to show that $\varphi(\overline{z}, \overline{\xi}, \overline{Y})$ is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers. For each $j \in [n]$ there exists $f_j \in C_{\text{s.a.}}(X)$ and an $\mathscr{L}^{C_{\text{s.a.}}(X)\text{-mod}}$-term $s'_j(\overline{z})$ such that $s_j(\overline{z}, x) = s'_j(\overline{z}) + f_j \cdot x$, and since $[-]_{\beta_j} : C_{\text{s.a.}}(X) \twoheadrightarrow \mathcal{O}_R$ is a ring homomorphism, $\varphi(\overline{z}, \overline{\xi}, \overline{Y})$ is equivalent to the formula (with germ parameters $[f_j]_{\beta_j} \in \mathcal{O}_R$ for $j \in [n]$)

$$\exists Y'_1 \dots Y'_n \left(\bigwedge_{i=1}^{n} Y_j = [s'_j(\overline{z})]_{\beta_j} + [f_j]_{\beta_j} Y'_j \wedge \underbrace{\exists x \left[\bigwedge_{i=1}^{m} \xi_i = \{t_i(\overline{z}, x) \geq 0\} \wedge \bigwedge_{i=1}^{n} Y'_j = [x]_{\beta_j} \right]}_{\varphi'(\overline{z}, \overline{\xi}, \overline{Y'})} \right).$$

The formula $\varphi'(\overline{z}, \overline{\xi}, \overline{Y'})$ is of the form (✽$_1$), therefore it is equivalent to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers by assumption, from which it follows that so is $\varphi(\overline{z}, \overline{\xi}, \overline{Y})$, as required. □

Lemma 3.2.9. *Suppose that that every $\mathscr{L}_X$-formula of the form*

$$\exists x \left[\bigwedge_{i=1}^{m} \xi_i = \{f_i \cdot x \geq t_i(\overline{z})\} \wedge \bigwedge_{j=1}^{n} Y_j = [x]_{\beta_j} \right] \qquad (✽_2)$$

is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers, where

(i) *x is a home variable;*

(ii) *$\xi_1, \dots, \xi_m$ are space variables and $Y_1, \dots, Y_n$ are germ variables;*

(iii) *$t_1(\overline{z}), \dots, t_m(\overline{z})$ are $\mathscr{L}^{C_{\text{s.a.}}(X)\text{-mod}}$-terms;*

(iv) $f_i \in C_{\text{s.a.}}(X)$ *are scalar functions such that* $f_i \geq 0$ *or* $f_i \leq 0$ *for all* $i \in [m]$*; and*

(v) $\beta_1, \ldots, \beta_n$ *are pairwise distinct half-branches of* X.

Then every $\mathscr{L}_X^{\text{const}}$*-formula is equivalent modulo* $\mathscr{M}_X^{\text{const}}$ *to an* $\mathscr{L}_X^{\text{const}}$*-formula without home quantifiers.*

Proof. Let $\varphi(\overline{z}, \overline{\xi}, \overline{Y})$ be an $\mathscr{L}_X$-formula of the form ($\clubsuit_1$); by Lemma 3.2.8 it suffices to show that $\varphi(\overline{z}, \overline{\xi}, \overline{Y})$ is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers. Suppose that there exists $j_0 \in [n]$ and $\varnothing \neq S \subseteq [n] \setminus \{j_0\}$ such that $\beta_{j_0} = \beta_j$ for all $j \in S$; then $\varphi(\overline{z}, \overline{\xi}, \overline{Y})$ is equivalent to

$$\bigwedge\!\!\!\bigwedge_{j \in S} Y_j = Y_{j_0} \wedge\!\!\!\wedge \exists x \left[\bigwedge\!\!\!\bigwedge_{i=1}^{m} \xi_i = \{t_i(\overline{z}, x) \geq 0\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{j \notin S} Y_j = [x]_{\beta_j} \right],$$

from which it follows that it can be assumed that all half-branches β_j appearing as germ function symbols $[-]_{\beta_j}$ in $\varphi(\overline{z}, \overline{\xi}, \overline{Y})$ are pairwise distinct. For each $i \in [n]$ there exists $f_i \in C_{\text{s.a.}}(X)$ and an $\mathscr{L}^{C_{\text{s.a.}}(X)\text{-mod}}$-term $t_i'(\overline{z})$ such that $t_i(\overline{z}, x) = -t_i'(\overline{z}) + f_i \cdot x$, therefore $\{t_i(\overline{z}, x) \geq 0\} = \{f_i \cdot x \geq t_i'(\overline{z})\}$. Note that for each $i \in [n]$, the formula $\xi_i = \{f_i \cdot x \geq t_i'(\overline{z})\}$ is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to the conjunction of the formulas (with space parameters $\{f_i \geq 0\}, \{f_i \leq 0\} \in L_X$)

(i) $\xi_i \sqcap \{f_i \geq 0\} = \{(f_i \vee 0) \cdot x \geq t_i(\overline{z})\} \sqcap \{f_i \geq 0\}$, and

(ii) $\xi_i \sqcap \{f_i \leq 0\} = \{(f_i \wedge 0) \cdot x \geq t_i(\overline{z})\} \sqcap \{f_i \leq 0\}$.

It follows that $\varphi(\overline{z}, \overline{\xi}, \overline{Y})$ is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to the $\mathscr{L}_X^{\text{const}}$-formula

$$\exists \overline{\xi^+} \exists \overline{\xi^-} \Bigg(\bigwedge\!\!\!\bigwedge_{i=1}^{m} \xi_i \sqcap \{f_i \geq 0\} = \xi_i^+ \sqcap \{f_i \geq 0\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{i=1}^{m} \xi_i \sqcap \{f_i \leq 0\} = \xi_i^- \sqcap \{f_i \leq 0\} \wedge\!\!\!\wedge$$
$$\psi \left\{ \begin{array}{l} \exists x \left[\bigwedge\!\!\!\bigwedge_{i=1}^{m} \xi_i^+ = \{(f_i \vee 0) \cdot x \geq t_i'(\overline{z})\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{i=1}^{m} \xi_i^- = \{(f_i \wedge 0) \cdot x \geq t_i'(\overline{z})\} \wedge\!\!\!\wedge \right. \\ \qquad \left. \bigwedge\!\!\!\bigwedge_{i=1}^{n} Y_j = [x]_{\beta_j} \right] \Bigg). \end{array} \right.$$

The formula $\psi(\overline{z}, \overline{\xi^+}, \overline{\xi^+}, \overline{Y})$ is of the form ($\clubsuit_2$), therefore it is equivalent to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers by assumption, from which it follows that so is $\varphi(\overline{z}, \overline{\xi}, \overline{Y})$, as required. $\square$

3.3 Elimination steps

3.3.1 Eliminating home quantifiers in formulas ($\bigstar_1$)

Lemma 3.3.1. *Let $f, g \in C_{\text{s.a.}}(X)$, $F_1, \dots, F_n \in \mathcal{O}_R$, and $\beta_1, \dots, \beta_n$ be half-branches of X at $b_1, \dots, b_n$, respectively. Let also $\{a_1, \dots, a_m\} := \partial_X(\{f = 0\})$. The following are equivalent:*

(i) *There exists $h \in C_{\text{s.a.}}(X)$ such that $g = f \cdot h$ and $F_l = [h]_{\beta_l}$ for all $l \in [n]$.*

(ii) *$\{f = 0\} \subseteq \{g = 0\}$, $[g]_{\beta_l} = [f]_{\beta_l} F_l$ for all $l \in [n]$, and there exists $h_0 \in C_{\text{s.a.}}(X)$ and $\varepsilon > 0$ such that $\bigcup_{i \in [m]} \overline{B}_\varepsilon(a_i) \subseteq \{g = f \cdot h_0\}$ and $F_l = [h_0]_{\beta_l}$ for all $l \in [n]$.*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious, so suppose that (ii) holds and let $h_0 \in C_{\text{s.a.}}(X)$ and $\varepsilon > 0$ witness this. Since $[g]_{\beta_l} = [f]_{\beta_l} F_l = [f]_{\beta_l} [h_0]_{\beta_l} = [f \cdot h_0]_{\beta_l}$, for each $l \in [n]$ there exists a curve interval C_l of β_l such that $C_l \subseteq \{g = f \cdot h_0\}$ by Proposition 2.3.35 (iv). Define $D_1 := \bigcup_{i \in [m]} \overline{B}_\varepsilon(a_i) \cup \bigcup_{l \in [n]} C_l$ and $h_1 := h_{0 \restriction D_1}$, noting that $D_1 \subseteq \{g_{\restriction D_1} = f_{\restriction D_1} \cdot h_1\}$. It now suffices to find $h \in C_{\text{s.a.}}(X)$ such that $h_{\restriction D_1} = h_1$ and $g = f \cdot h$, since for any such h one has $h_{\restriction C_l} = h_{1 \restriction C_l} = h_{0 \restriction C_l}$ and hence $[h]_{\beta_l} = [h_0]_{\beta_l} = F_l$ for all $l \in [n]$.

Let $D_2 := D_1 \cup \{f = 0\}$. By the semi-algebraic Tietze extension theorem (see Theorem 2.3.17) there exists $h_2 : D_2 \longrightarrow R$ continuous and semi-algebraic such that $h_{2 \restriction D_1} = h_1$. Since $\{f = 0\} \subseteq \{g = 0\}$, it follows by choice of h_2 that $D_2 \subseteq \{g_{\restriction D_2} = f_{\restriction D_2} \cdot h_2\}$. Let now $D_3 := \mathrm{cl}_X(X \setminus D_2)$. Since $\mathrm{cl}_X(X \setminus D_2) = X \setminus \mathrm{int}_X(D_2)$, it follows that $D_3 \cap \{f = 0\} = \varnothing$: indeed, if $a \in \{f = 0\}$, then either $a \in \mathrm{int}_X(\{f = 0\})$, in which case $a \in \mathrm{int}_X(D_2)$, or $a \in \partial_X(\{f = 0\})$, in which case $a = a_i$ for some $i \in [m]$ and thus $a \in \mathrm{int}_X(D_1) \subseteq \mathrm{int}_X(D_2)$. In particular, the function $h_3 : D_3 \longrightarrow R$ defined by $h_3(x) := g(x)/f(x)$ is well-defined, continuous, and semi-algebraic. To conclude, note that $X = D_2 \cup D_3$ and define $h : X \longrightarrow R$ by

$$h(x) := \begin{cases} h_2(x) & \text{if } x \in D_2 \\ h_3(x) & \text{if } x \in D_3. \end{cases}$$

If $x \in D_2 \cap D_3$, then $f(x) \neq 0$ by the above and $h_3(x) = g(x)/f(x)$ by definition of h_3, and also $g(x)/f(x) = (f(x)h_2(x))/f(x) = h_2(x)$ since $D_2 \subseteq \{g_{\restriction D_2} = f_{\restriction D_2} \cdot h_2\}$. It follows that $h \in C_{\text{s.a.}}(X)$ and it satisfies $h_{\restriction D_1} = h_1$ and $g = f \cdot h$ by construction, as required. $\square$

Lemma 3.3.2. *Let $f, g \in C_{\text{s.a.}}(X)$, $F_1, \dots, F_n \in \mathcal{O}_R$, and $\beta_1, \dots, \beta_n$ be pairwise distinct half-branches of X centred at $b_1, \dots, b_n$, respectively. Let also $\{a_1, \dots, a_m\} := \partial_X(\{f = 0\})$, and for each $i \in [m]$, let $\gamma_{i,1}, \dots, \gamma_{i,n_i}$ be all the half-branches of X at a_i. The following are equivalent:*

(I) *There exists $h_0 \in C_{\text{s.a.}}(X)$ and $\varepsilon > 0$ such that $\bigcup_{i\in[m]} \overline{B}_\varepsilon(a_i) \subseteq \{g = f \cdot h_0\}$ and $F_l = [h_0]_{\beta_l}$ for all $l \in [n]$.*

(II) *The following conditions hold:*

(i) *For all $l_1, l_2 \in [n]$, if $b_{l_1} = b_{l_2}$, then $F_{l_1} - F_{l_2} \in \mathfrak{m}$.*

(ii) *There exist $H_{ij} \in \mathcal{O}_R$ for all $i \in [m]$ and all $j \in [n_i]$ such that the following conditions hold:*

(a) *For all $i \in [m]$, and all $j_1, j_2 \in [n_i]$, $H_{ij_1} - H_{ij_2} \in \mathfrak{m}$.*

(b) *For all $i \in [m]$ and all $j \in [n_i]$, $[g]_{\gamma_{i,j}} = [f]_{\gamma_{i,j}} H_{ij}$.*

(c) *For all $i \in [m]$, all $j \in [n_i]$, and all $l \in [n]$, if $\gamma_{i,j} = \beta_l$, then $H_{ij} = F_l$.*

Proof. (I) $\Rightarrow$ (II). If $l_1, l_2 \in [n]$ and $b_{l_1} = b_{l_2}$, then β_{l_1} and β_{l_2} are two half-branches of X centred at the same point, therefore $F_{l_1} - F_{l_2} = [h_0]_{\beta_{l_1}} - [h_0]_{\beta_{l_2}} \in \mathfrak{m}$ follows by Proposition 2.3.35 (ii). Set $H_{ij} := [h_0]_{\gamma_{i,j}}$ for all $i \in [n]$ and all $j \in [n_i]$. Then (a) holds by the same argument as above, and (c) holds trivially. To prove that (b) holds, pick $i \in [m]$ and $j \in [n_i]$. Then, choosing a smaller $\varepsilon > 0$ if necessary, there exists a connected component U of $\overline{B}_\varepsilon(a_i) \setminus \{a_i\}$ such that $C := \text{cl}_X(U)$ is a curve interval of $\gamma_{i,j}$; such curve interval satisfies $C \subseteq \{g = f \cdot h_0\}$, which implies by Proposition 2.3.35 (iv) that $[g]_{\gamma_{i,j}} = [f \cdot h_0]_{\gamma_{i,j}} = [f]_{\gamma_{i,j}} [h_0]_{\gamma_{i,j}} = [f]_{\gamma_{i,j}} H_{ij}$, as required.

(II) $\Rightarrow$ (I). Let $H_{ij} \in \mathcal{O}_R$ be as in item (ii). For each $l \in [n]$, let $f_l \in C_{\text{s.a.}}(X)$ be such that $F_l = [f_l]_{\beta_l}$, and for all $i \in [m]$ and all $j \in [n_i]$ let $h_{ij} \in C_{\text{s.a.}}(X)$ be such that $H_{ij} = [h_{ij}]_{\gamma_{i,j}}$. In particular, by Proposition 2.3.35 (iv) items (b) and (c) respectively imply that:

(b') For all $i \in [m]$ and all $j \in [n_i]$, there exists a curve interval C_{ij} of γ_{ij} such that $C_{ij} \subseteq \{g = f \cdot h_{ij}\}$.

(c') For all $i \in [m]$, all $j \in [n_i]$, and all $l \in [n]$, if $\gamma_{ij} = \beta_l$, then there exists a curve interval C_{ij} of $\gamma_{ij,} = \beta_l$ such that $C_{ij} \subseteq \{h_{ij} = f_l\}$.

It follows that by choosing small enough curve intervals, there exist curve intervals C_{ij} of $\gamma_{i,j}$ and curve intervals C_l of β_l such that all the following conditions hold:

1. For all $i \in [m]$ and all $j \in [n_i]$, $C_{ij} \subseteq \{g = f \cdot h_{ij}\}$.

2. For all $i \in [m]$, all $j \in [n_i]$, and all $l \in [n]$, if $\gamma_{i,j} = \beta_l$, then $C_{ij} = C_l$ and $C_{ij} = C_l \subseteq \{h_{ij} = f_l\}$.

3. If $l_1, l_2 \in [n]$ and $b_{l_1} = b_{l_2}$, then $C_{l_1} \cap C_{l_2} = \{b_{l_1}\} = \{b_{l_2}\}$.

4. If $l_1, l_2 \in [n]$ and $b_{l_1} \neq b_{l_2}$, then $C_{l_1} \cap C_{l_2} = \varnothing$.

5. For all $i \in [m]$, if $j_1, j_2 \in [n_i]$ are such that $j_1 \neq j_2$, then $C_{ij_1} \cap C_{ij_2} = \{a_i\}$.

6. For all $i, i' \in [m]$, if $i \neq i'$, then $C_{ij} \cap C_{i'j'} = \varnothing$ for all $j \in [n]$ and all $j' \in [n']$.

7. For all $i \in [m]$ and all $l \in [n]$, if $a_i \neq b_l$, then $C_{ij} \cap C_l = \varnothing$ for all $j \in [n_i]$.

Let $D_i := \bigcup_{j \in [n_i]} C_{ij}$ for each $i \in [m]$, and let $E := \bigcup_{l \in [n]} C_l$. Define $h_i : D_i \longrightarrow R$ by $h_i(x) := h_{ij}(x)$ if $x \in C_{ij}$ and $h_E : E \longrightarrow R$ by $h_E(x) := f_l(x)$ if $x \in C_l$. For each $i \in [m]$, if $x \in C_{ij_1} \cap C_{ij_2}$ for some $j_1, j_2 \in [n_i]$ such that $j_1 \neq j_2$, then $x = a_i$ by item 5 , and $h_{ij_1}(a_i) = h_{ij_2}(a_i)$ by item (a) and Proposition 2.3.35 (ii), therefore $h_i : D_i \longrightarrow R$ is continuous and semi-algebraic. Similarly, if $x \in C_{l_1} \cap C_{l_2}$ for some $l_1, l_2 \in [n]$ such that $l_1 \neq l_2$, then $x = b_{l_1} = b_{l_2}$ by items 3 and 4, and $f_{l_1}(b_{l_1}) = f_{l_2}(b_{l_2})$ by item (i) and Proposition 2.3.35 (ii), therefore $h_E : E \longrightarrow R$ is continuous and semi-algebraic. Set $D := \bigcup_{i \in m} D_i$. Since $D_i \cap D_{i'} = \varnothing$ for all $i, i' \in [m]$ such that $i \neq i'$ by item 6, it follows that $h_D : D \longrightarrow R$ defined by $h_D(x) := h_i(x)$ if $x \in D_i$ is continuous and semi-algebraic.

Define $h' : D \cup E \longrightarrow R$ by $h'(x) := h_D(x)$ if $x \in D$ and $h'(x) := h_E(x)$ if $x \in E$. If $x \in D \cap E$, then there exists $i \in [m]$, $j \in [n_i]$, and $l \in [n]$ such that $x \in C_{ij} \cap C_l$; by item 7 it follows that $a_i = b_l$, and since $\gamma_{i,1}, \ldots, \gamma_{i,n_i}$ are all the half-branches of X at a_i and β_l is a half-branch of X at b_l, there exists $j' \in [n_i]$ such that $\gamma_{i,j'} = \beta_l$. If $j' = j$, then $\gamma_{i,j} = \beta_l$, therefore $C_{ij} = C_l$ and $h_D(x) = h_{ij}(x) = f_l(x) = h_E(x)$ by item 2. If $j' \neq j$, then $C_{ij'} = C_l$ and $h_{ij'}(x) = f_l(x) = h_E(x)$ by item 2; moreover, $C_{ij'} = C_l$ and $x \in C_{ij} \cap C_l$ together imply $x = a_i$ by item 5, therefore $h_D(x) = h_{ij}(a_i) = h_{ij'}(a_i) = h_{ij'}(x) = h_E(x)$, where $h_{ij}(a_i) = h_{ij'}(a_i)$ holds by (a) and Proposition 2.3.35 (ii). Altogether, this shows that $h' : D \cup E \longrightarrow R$ is continuous and semi-algebraic.

Pick any $h_0 \in C_{\text{s.a.}}(X)$ extending h' (see Theorem 2.3.17). For each $i \in [m]$, the set D_i is a neighbourhood of a_i in X by construction, therefore there exists $\varepsilon > 0$ such that $\overline{B}_\varepsilon(a_i) \subseteq D_i$ for all $i \in [m]$. Since h_0 extends h_{ij} on each C_{ij} and $\bigcup_{i\in[m]} \overline{B}_\varepsilon(a_i) \subseteq \bigcup_{i\in[m]} \bigcup_{j\in[n_i]} C_{ij}$ by construction, it follows by item 1 that $\bigcup_{i\in[m]} \overline{B}_\varepsilon(a_i) \subseteq \{g = f \cdot h_0\}$; since h_0 extends f_l on each C_l, it follows that $C_l \subseteq \{h_0 = f_l\}$, hence $[h_0]_{\beta_l} = [f_l]_{\beta_l} = F_l$ by Proposition 2.3.35 (iv), as required. □

The condition (II) (ii) (c) in Lemma 3.3.2 involves an equality of half-branches of X. Two half-branches β_1 and β_2 of X are equal if and only if their corresponding prime filters $\mathfrak{f}_{\beta_1}$ and $\mathfrak{f}_{\beta_2}$ in the lattice L_X are equal (see Definition 2.3.33). In what follows it will be shown that each prime filter of L_X of the form $\mathfrak{f}_\beta$ for a half-branch β of X is definable in L_X with parameters, making thus the statement $\beta_1 = \beta_2$ expressible in the lattice L_X.

Lemma 3.3.3. *For every $a \in X$, the set of curve intervals of X at a is $\{\boldsymbol{a}\}$-definable in the $\mathscr{L}^{\text{lat}}(\top, \bot)$-structure L_X, where $\boldsymbol{a} := \{a\} \in L_X$.*

Proof. Note first that the set of atoms of L_X is defined by the formula (in the free variable ζ)

$$\zeta \neq \bot \mathbin{\curlywedge} \forall \xi[(\xi \neq \bot \mathbin{\curlywedge} \xi \sqsubseteq \zeta) \to \xi = \zeta],$$

and the set of those $C \in L_X$ which are semi-algebraically connected in X is defined by the formula (in the free variable ζ)

$$\forall \xi_1 \xi_2[(\xi_1 \neq \bot \neq \xi_2 \mathbin{\curlywedge} \xi_1 \sqcap \xi_2 = \bot \mathbin{\curlywedge} \zeta = \xi_1 \sqcup \xi_2) \to (\zeta = \xi_1 \mathbin{\curlyvee} \zeta = \xi_2)].$$

It follows from Lemma 2.3.31 and its proof that the set of those $C \in L_X$ which are curve intervals of X at a is defined by the formula (with parameter $\boldsymbol{a}$ and in the free variable ζ) expressing: "ζ is semi-algebraically connected, $\boldsymbol{a} \sqsubseteq \zeta$, and there exists an atom $\boldsymbol{b} \in L_X$ contained in ζ and distinct from $\boldsymbol{a}$ such that ζ is minimal in L_X amongst those semi-algebraically connected $D \in L_X$ with $\boldsymbol{a}, \boldsymbol{b} \sqsubseteq D$". □

Remark 3.3.4. The proof of Lemma 3.3.3 shows that the set of all curve intervals of X is an $\varnothing$-definable subset of L_X. In [Tre17, Section 4] is given another $\mathscr{L}^{\text{lat}}(\top, \bot)$-formula without parameters which defines the set of all those $C \in L_X$ which are curve intervals of X.

Definition 3.3.5. Fix a curve interval C_β of β for each $a \in X$ and each half-branch β of X at a. Define the $\mathscr{L}^{\text{lat}}(\top, \bot)$-formula $\mathtt{PrimF}_\beta(\zeta)$ with parameters $\{\boldsymbol{a}, C_\beta\}$ to be the one expressing: "there exists a curve interval C of a contained in C_β such that $C \sqsubseteq \zeta$."

Lemma 3.3.6. *Let $a \in X$ and β be a half-branch of X at a. The formula $\mathtt{PrimF}_\beta(\zeta)$ with parameters $\{\boldsymbol{a}, C_\beta\}$ defines the prime filter $\mathfrak{f}_\beta = \{C \in L_X \mid C \in \beta\}$ in L_X.*

Proof. Let $D \in L_X$, suppose first that $L_X \models \mathtt{PrimF}_\beta(D)$, and let $C \in L_X$ be a curve interval witnessing this. Then C and C_β have the same germ at a, therefore $C \in \beta$ and hence $C \in \mathfrak{f}_\beta$; since $C \subseteq D$ and $\mathfrak{f}_\beta$ is a filter of L_X, $D \in \mathfrak{f}_\beta$ follows. Conversely, if $D \in \mathfrak{f}_\beta$ then $D \in \beta$, and since β is a half-branch of X at a, there exists a curve interval C_0 of β such that $C_0 \subseteq D$ by Lemma 2.3.28, from which it follows that there exists a curve interval C of β contained in $C_0 \cap C_\beta$, and such C witnesses $L_X \models \mathtt{PrimF}_\beta(D)$. $\square$

Corollary 3.3.7. *Let $f \in C_{\text{s.a.}}(X)$ and let $\beta_1, \ldots, \beta_n$ be pairwise distinct half-branches of X centred at $b_1, \ldots, b_n$, respectively. Let also $\{a_1, \ldots, a_m\} := \partial_X(\{f = 0\})$, and for each $i \in [m]$, let $\gamma_{i1}, \ldots, \gamma_{in_i}$ be all the half-branches of X at a_i. The $\mathscr{L}_X$-formula (in free variables $y, Y_l, \ldots, Y_n$)*

$$\exists x \left(y = f \cdot x \curlywedge \bigcurlywedge_{l=1}^{n} Y_l = [x]_{\beta_l} \right) \tag{3.9}$$

is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to the formula

$$\exists \zeta \exists \overline{Y'} \exists \overline{Z} \Bigg(\psi(\zeta, \overline{Y}, \overline{Y'}, \overline{Z}) \curlywedge \zeta = \{y = 0\} \curlywedge \bigcurlywedge_{l=1}^{n} Y'_l = [y]_{\beta_l} \curlywedge \bigcurlywedge_{i \in [m]} \bigcurlywedge_{j \in [n_i]} Z_{ij} = [y]_{\gamma_{ij}} \Bigg) \tag{3.10}$$

where $\psi(\zeta, \overline{Y}, \overline{Y'}, \overline{Z})$ is the $\mathscr{L}_X^{\text{const}}$-formula

$$\{f = 0\} \sqsubseteq \zeta \curlywedge \bigcurlywedge_{l \in [n]} Y'_l = [f]_{\beta_l} Y_l \curlywedge \bigcurlywedge_{l_1, l_2 \in [n]} \boldsymbol{b}_{l_1} = \boldsymbol{b}_{l_2} \to \mathtt{m}(Y_{l_1} - Y_{l_2}) \curlywedge$$
$$\exists \overline{Z'} \Bigg(\bigcurlywedge_{i \in [m]} \bigcurlywedge_{j_1, j_2 \in [n_i]} \mathtt{m}(Z'_{ij_1} - Z'_{ij_2}) \curlywedge \bigcurlywedge_{i \in [m]} \bigcurlywedge_{j \in [n_i]} Z_{ij} = [f]_{\gamma_{ij}} Z'_{ij} \curlywedge \bigcurlywedge_{i \in [m]} \bigcurlywedge_{j \in [n_i]} \bigcurlywedge_{l \in [n]} \forall \xi (\mathtt{PrimF}_{\gamma_{ij}}(\xi) \leftrightarrow \mathtt{PrimF}_{\beta_l}(\xi)) \to Z'_{ij} = Y_l \Bigg).$$

Proof. The formula (3.10) is equivalent to the formula

$$\{f=0\} \sqsubseteq \{y=0\} \wedge \bigwedge_{l\in[n]} [y]_{\beta_l} = [f]_{\beta_l} Y_l \wedge \bigwedge_{l_1,l_2\in[n]} \boldsymbol{b}_{l_1} = \boldsymbol{b}_{l_2} \to \mathtt{m}(Y_{l_1} - Y_{l_2}) \wedge$$
$$\exists \overline{Z'} \Bigg(\bigwedge_{i\in[m]} \bigwedge_{j_1,j_2\in[n_i]} \mathtt{m}(Z'_{ij_1} - Z'_{ij_2}) \wedge$$
$$\bigwedge_{i\in[m]} \bigwedge_{j\in[n_i]} [y]_{\gamma_{ij}} = [f]_{\gamma_{ij}} Z'_{ij} \wedge$$
$$\bigwedge_{i\in[m]} \bigwedge_{j\in[n_i]} \bigwedge_{l\in[n]} \forall \xi (\mathtt{PrimF}_{\gamma_{ij}}(\xi) \leftrightarrow \mathtt{PrimF}_{\beta_l}(\xi)) \to Z'_{ij} = Y_l \Bigg).$$

and this latter formula is clearly equivalent to (3.9) by Lemma 3.3.1 and Lemma 3.3.2. □

Proposition 3.3.8. *Let $f \in C_{\text{s.a.}}(X)$ and let $\beta_1, \dots, \beta_n$ be pairwise distinct half-branches of X. The $\mathscr{L}_X$-formula (in free variables $y, Y_l, \dots, Y_n$)*

$$\exists x \left(y = f \cdot x \wedge \bigwedge_{l=1}^{n} Y_l = [x]_{\beta_l} \right) \qquad (\bigstar_1)$$

is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to a formula of the form

$$\exists \zeta \exists \overline{Y'} \left(\psi(\zeta, \overline{Y}, \overline{Y'}) \wedge \zeta = \{y=0\} \wedge \bigwedge_{l=1}^{n'} Y'_l = [y]_{\beta'_l} \right), \qquad (3.11)$$

where $\psi(\zeta, \overline{Y}, \overline{Y'})$ is a Boolean combination of space formulas with parameters and germ formulas with parameters, and $\beta'_1, \dots, \beta'_{n'}$ are pairwise distinct half-branches.

Proof. For each $l \in [n]$, let b_l be the point at which β_l is centred. Let $\{a_1, \dots, a_m\} := \partial_X(\{f=0\})$, and for each $i \in [m]$, let $\gamma_{i1}, \dots, \gamma_{in_i}$ be all the half-branches of X at a_i. By Corollary 3.3.7, the formula ($\bigstar_1$) is equivalent to

$$\exists \zeta \exists \overline{Y'} \exists \overline{Z} \Bigg(\psi(\zeta, \overline{Y}, \overline{Y'}, \overline{Z}) \wedge \zeta = \{y=0\} \wedge \bigwedge_{l=1}^{n} Y'_l = [y]_{\beta_l} \wedge$$
$$\bigwedge_{i\in[m]} \bigwedge_{j\in[n_i]} Z_{ij} = [y]_{\gamma_{ij}} \Bigg), \qquad (3.12)$$

where $\psi(\zeta, \overline{Y}, \overline{Y'}, \overline{Z})$ is the $\mathscr{L}_X^{\text{const}}$-formula defined as in Corollary 3.3.7. Let S be the set of triples (i, j, l) such that $\gamma_{ij} = \beta_l$ and T be the set of those $l \in [n]$ for which there does not exist $i \in [m]$ and $j \in [n_i]$ such that $(i, j, l) \in S$; then (3.12) is equivalent to

$$\exists\zeta\exists\overline{Y'}\exists\overline{Z}\Bigg(\overbrace{\psi(\zeta,\overline{Y},\overline{Y'},\overline{Z}) \curlywedge \bigcurlywedge_{(i,j,l)\in S} Z_{ij} = Y'_l}^{(*)} \curlywedge \zeta = \{y=0\} \curlywedge \bigcurlywedge_{l\in T} Y'_l = [y]_{\beta_l} \curlywedge \bigcurlywedge_{i\in[m]} \bigcurlywedge_{j\in[n_i]} Z_{ij} = [y]_{\gamma_{ij}}\Bigg), \tag{3.13}$$

and γ_{ij} and β_l are pairwise distinct half-branches of X for all $i \in [m]$, $j \in [n_i]$ and $l \in T$. The formula $(*)$ is a $(\mathscr{L}_X^{\text{const}})_{\restriction\Sigma}$-formula, therefore it is equivalent to a Boolean combination of space formulas with parameters and germ formulas with parameters by Lemma 2.1.6 (see also the proof of Lemma 3.2.5), from which it follows that (3.13) is equivalent to a formula of the form (3.11). □

3.3.2 Eliminating home quantifiers in formulas (★$_2$)

Lemma 3.3.9. *Let $F_1, \ldots, F_n \in \mathcal{O}_R$, and $\beta_1, \ldots, \beta_n$ be pairwise distinct half-branches of X at $b_1, \ldots, b_n$, respectively. Suppose that for all $l_1, l_2 \in [n]$, if $b_{l_1} = b_{l_2}$, then $F_{l_1} - F_{l_2} \in \mathfrak{m}$. Then for all $h_1, \ldots, h_n \in C_{\text{s.a.}}(X)$ such that $[h_l]_{\beta_l} = F_l$ there exist curve intervals $C_1, \ldots, C_n$ of $\beta_1, \ldots, \beta_n$ (respectively) and $h \in C_{\text{s.a.}}(X)$ such that $C_l \subseteq \{h = h_l\}$ for all $l \in [n]$.*

Proof. For each $l \in [n]$, pick $h_l \in C_{\text{s.a.}}(X)$ such that $F_l = [h_l]_{\beta_l}$ and a curve interval C_l of β_l. Since $\beta_1, \ldots, \beta_n$ are pairwise distinct, one may choose the curve intervals C_l such that $C_{l_1} \cap C_{l_2} = \{b_{l_1}\} = \{b_{l_2}\}$ if $b_{l_1} = b_{l_2}$, and $C_{l_1} \cap C_{l_2} = \varnothing$ if $b_{l_1} \neq b_{l_2}$ for all $l_1, l_2 \in [n]$ such that $l_1 \neq l_2$. Define $h_0 : \bigcup_{l\in[n]} C_l \longrightarrow R$ by $h_0(x) := h_l(x)$ if $x \in C_l$. If $l_1, l_2 \in [n]$ are such that $l_1 \neq l_2$ and $x \in C_{l_1} \cap C_{l_2}$, then $x = b_{l_1} = b_{l_2}$ by choice of the curve intervals C_l, therefore

$$h_{l_1}(x) = h_{l_1}(b_{l_1}) = h_{l_1}(b_{l_2}) \overset{(*)}{=} h_{l_2}(b_{l_2}) = h_{l_2}(x),$$

where $(*)$ follows from $[h_{l_1}]_{\beta_{l_1}} - [h_{l_2}]_{\beta_{l_2}} = F_{l_1} - F_{l_2} \in \mathfrak{m}$ and Proposition 2.3.35 (ii). This shows that $h_0 : \bigcup_{l\in[n]} C_l \longrightarrow R$ is continuous and semi-algebraic. Let $h \in C_{\text{s.a.}}(X)$ be any continuous semi-algebraic extension of h_0; then h satisfies $C_l \subseteq \{h = h_l\}$ for all $l \in [n]$, as required. □

Lemma 3.3.10. *Let $f, g \in C_{\text{s.a.}}(X)$, $D_1, D_2 \in L_X$, $F_1, \ldots, F_n \in \mathcal{O}_R$, and $\beta_1, \ldots, \beta_n$ be pairwise distinct half-branches of X at $b_1, \ldots, b_n$. The following are equivalent:*

(I) *There exists* $h \in C_{\text{s.a.}}(X)$ *such that*

$$D_1 \subseteq \{h \geq f\} \curlywedge D_2 \subseteq \{h \leq g\} \curlywedge \bigcurlywedge_{l=1}^{n} F_l = [h]_{\beta_l}.$$

(II) *The following conditions hold:*

(i) *For all* $l_1, l_2 \in [n]$, *if* $b_{l_1} = b_{l_2}$, *then* $F_{l_1} - F_{l_2} \in \mathfrak{m}$.

(ii) $D_1 \cap D_2 \subseteq \{f \leq g\}$.

(iii) *For all* $l \in [n]$, *the following conditions hold:*

(a) *If* $D_1 \in \mathfrak{f}_{\beta_l}$, *then* $F_l \geq [f]_{\beta_l}$.

(b) *If* $D_1 \notin \mathfrak{f}_{\beta_l}$ *and* $b_l \in D_1$, *then* $F_l/\mathfrak{m} \geq [f]_{\beta_l}/\mathfrak{m}$.

(c) *If* $D_2 \in \mathfrak{f}_{\beta_l}$, *then* $F_l \leq [g]_{\beta_l}$.

(d) *If* $D_2 \notin \mathfrak{f}_{\beta_l}$ *and* $b_l \in D_2$, *then* $F_l/\mathfrak{m} \leq [g]_{\beta_l}/\mathfrak{m}$.

Proof. (I) ⇒ (II). Pick $l_1, l_2 \in [n]$ and suppose that $b_{l_1} = b_{l_2}$; then β_{l_1} and β_{l_2} are two half-branches of X centred at the same point, therefore $F_{l_1} - F_{l_2} = [h]_{\beta_{l_1}} - [h]_{\beta_{l_2}} \in \mathfrak{m}$ by Proposition 2.3.35 (ii) and thus (i) holds. Also, $D_1 \cap D_2 \subseteq \{h \geq f\} \cap \{h \leq g\} \subseteq \{f \leq g\}$, therefore (ii) holds. Fix now $l \in [n]$. If $D_1 \in \mathfrak{f}_{\beta_l}$, then by Lemma 2.3.28 there exists a curve interval C of β_l such that $C \subseteq D_1$, therefore $C \subseteq \{h \geq f\}$ and thus $F_l = [h]_{\beta_l} \geq [f]_{\beta_l}$ by Proposition 2.3.35 (iv); this shows that (a) holds, and (c) holds analogously. If $D_1 \notin \mathfrak{f}_{\beta_l}$ and $b_l \in D_1$, then $b_l \in \{h \geq f\}$, therefore $h(b_l) \geq f(b_l)$, and thus $F_l/\mathfrak{m} = [h]_{\beta_l}/\mathfrak{m} \geq [f]_{\beta_l}/\mathfrak{m}$ by Proposition 2.3.35 (ii); this shows that (d) holds, and (d) holds analogously.

(II) ⇒ (I). For each $l \in [n]$, pick $h_l \in C_{\text{s.a.}}(X)$ such that $F_l = [h_l]_{\beta_l}$. By item (i) and Lemma 3.3.9 there exist curve intervals $C'_1, \ldots, C'_n$ of $\beta_1, \ldots, \beta_n$ (respectively) and $h' \in C_{\text{s.a.}}(X)$ such that $C'_l \subseteq \{h' = h_l\}$ for all $l \in [n]$. It now suffices to prove that there exist curve intervals $C_1, \ldots, C_n$ of $\beta_1, \ldots, \beta_n$ (respectively) and $h \in C_{\text{s.a.}}(X)$ such that

$$D_1 \subseteq \{h \geq f\} \curlywedge D_2 \subseteq \{h \leq g\} \curlywedge \bigcurlywedge_{l \in [n]} C_l \subseteq \{h' = h\},$$

that is, such that

$$\left(D_1 \subseteq \{h \geq f\} \curlywedge \bigcurlywedge_{l\in[n]} C_l \subseteq \{h \geq h'\}\right) \curlywedge$$
$$\left(D_2 \subseteq \{h \leq g\} \curlywedge \bigcurlywedge_{l\in[n]} C_l \subseteq \{h \leq h'\}\right).$$

Indeed, if such $C_1, \dots, C_n$ and h exist, then $C'_l \cap C_l \in \beta_l$ for each $l \in [n]$, hence there exists a curve interval C''_l of β_l such that $C''_l \subseteq C'_l \cap C_l$ by Lemma 2.3.28, therefore $C''_l \subseteq \{h' = h_l\} \cap \{h' = h\} \subseteq \{h = h_l\}$, hence $[h]_{\beta_l} = [h_l]_{\beta_l} = F_l$ by Proposition 2.3.35 (iv). By Lemma 2.4.18, it suffices in turn to show that there exist curve intervals $C_1, \dots, C_n$ of $\beta_1, \dots, \beta_n$ such that all the conditions below hold:

1. There exists $h \in C_{\text{s.a.}}(X)$ such that $D_1 \subseteq \{h \geq f\}$ and $D_2 \subseteq \{h \leq g\}$.

2. For all $l \in [n]$ there exists $h \in C_{\text{s.a.}}(X)$ such that
$$D_1 \subseteq \{h \geq f\} \text{ and } C_l \subseteq \{h \leq h'\}.$$

3. For all $l \in [n]$ there exists $h \in C_{\text{s.a.}}(X)$ such that
$$C_l \subseteq \{h \geq h'\} \text{ and } D_2 \subseteq \{h \leq g\}.$$

4. For all $l_1, l_2 \in [n]$ there exists $h \in C_{\text{s.a.}}(X)$ such that $C_{l_1} \subseteq \{h \geq h'\}$ and $C_{l_2} \subseteq \{h \leq h'\}$.

Item 1 holds by item (ii) in the statement of the lemma and Proposition 2.4.25 (I), and item 4 holds trivially for any choice of curve intervals by taking $h := h'$. It therefore remains to show that there exist curve intervals $C_1, \dots, C_n$ of $\beta_1, \dots, \beta_n$ such that conditions 2 and 3 above hold. In turn, this is equivalent to showing that for all $l \in [n]$ there exists a curve interval C_l of β_l such that the following two conditions hold for C_l:

2'. There exists $h \in C_{\text{s.a.}}(X)$ such that $D_1 \subseteq \{h \geq f\}$ and $C_l \subseteq \{h \leq h'\}$.

3'. There exists $h \in C_{\text{s.a.}}(X)$ such that $C_l \subseteq \{h \geq h'\}$ and $D_2 \subseteq \{h \leq g\}$.

By Proposition 2.4.25 (I), it suffices to prove that for all $l \in [n]$ there exists a curve interval C_l of β_l such that

$$C_l \cap D_1 \subseteq \{f \leq h'\} \text{ and } C_l \cap D_2 \subseteq \{h' \leq g\}. \tag{$*$}$$

For what follows, recall that $C'_l \subseteq \{h' = h_l\}$ for all $l \in [n]$, so that in particular $[h']_{\beta_l} = [h_l]_{\beta_l} = F_l$. Fix $l \in [n]$; then exactly one of the following situations holds:

- $D_1 \in \mathfrak{f}_{\beta_l}$. In this case there exists a curve interval I of β_l such that $I \subseteq D_1 \cap \{f \leq h'\}$ by item (a); in particular $I \cap D_1 \subseteq \{f \leq h'\}$.
- $D_1 \notin \mathfrak{f}_{\beta_l}$ and $b_l \in D_1$. In this case there exists a curve interval I of β_l such that $I \cap D_1 = \{b_l\}$ and $b_l \in \{f \leq h'\}$ by item (b); in particular $I \cap D_1 \subseteq \{f \leq h'\}$.
- $b_l \notin D_1$. In this case exists a curve interval I of β_l such that $I \cap D_1 = \varnothing$; in particular $I \cap D_1 \subseteq \{f \leq h'\}$.

In either of the cases above there exists a curve interval I of β_l such that $I \cap D_1 \subseteq \{f \leq h'\}$. Similarly, there exists a curve interval J of β_l such that $J \cap D_2 \subseteq \{h' \leq g\}$ appealing this time to items (c) and (d). Since $I \cap J \in \beta_l$, there exists a curve interval C_l of β_l such that $C_l \subseteq I \cap J$; such curve interval satisfies $(*)$, and this concludes the proof. $\square$

Lemma 3.3.11. *Let $f, g \in C_{\text{s.a.}}(X)$, $D_1, D_2 \in L_X$, $F_1, \ldots, F_n \in \mathcal{O}_R$, and $\beta_1, \ldots, \beta_n$ be pairwise distinct half-branches of X at $b_1, \ldots, b_n$. The following are equivalent:*

(I) *There exists $h \in C_{\text{s.a.}}(X)$ such that*

$$\{h \leq f\} \subseteq D_1 \curlywedge D_2 \subseteq \{h \leq g\} \curlywedge \bigwedge_{l=1}^{n} F_l = [h]_{\beta_l}.$$

(II) *The following conditions hold:*

(i) *For all $l_1, l_2 \in [n]$, if $b_{l_1} = b_{l_2}$, then $F_{l_1} - F_{l_2} \in \mathfrak{m}$.*

(ii) $D_2 \cap \{g \leq f\} \subseteq D_1$.

(iii) *For all $l \in [n]$, the following conditions hold:*

(a) *If $F_l \leq [f]_{\beta_l}$, then $D_1 \in \mathfrak{f}_{\beta_l}$.*

(b) *If $F_l/\mathfrak{m} \leq [f]_{\beta_l}/\mathfrak{m}$, then $b_l \in D_1$.*

(c) *If $D_2 \in \mathfrak{f}_{\beta_l}$, then $F_l \leq [g]_{\beta_l}$.*

(d) *If $D_2 \notin \mathfrak{f}_{\beta_l}$ and $b_l \in D_2$, then $F_l/\mathfrak{m} \leq [g]_{\beta_l}/\mathfrak{m}$.*

(iv) *There exists $D'_1 \in L_X$ such that the following conditions hold:*

(a) $D_1' \cup D_1 = X$ *and* $D_1' \cap D_2 \subseteq \{f \leq g\}$.

(b) *For all* $l \in [n]$*, the following conditions hold:*

(a') *If* $D_1' \in \mathfrak{f}_{\beta_l}$*, then* $F_l \geq [f]_{\beta_l}$.

(b') *If* $D_1' \notin \mathfrak{f}_{\beta_l}$ *and* $b_l \in D_1'$*, then* $F_l/\mathfrak{m} \geq [f]_{\beta_l}/\mathfrak{m}$.

Proof. (I) $\Rightarrow$ (II). Item (i) holds using the same argument as in the proof of (I) $\Rightarrow$ (II) (i) in Lemma 3.3.10. Also, $D_2 \cap \{g \leq f\} \subseteq \{h \leq g\} \cap \{g \leq f\} \subseteq \{h \leq f\} \subseteq D_1$, therefore (ii) holds. Fix $l \in [n]$. If $[h]_{\beta_l} = F_l \leq [f]_{\beta_l}$, then there exists a curve interval C of β_l such that $C \subseteq \{h \leq f\} \subseteq D_1$ by Proposition 2.3.35 (iv), therefore $D_1 \in \mathfrak{f}_{\beta_l}$ and thus (iii) (a) holds. Similarly, if $[h]_{\beta_l}/\mathfrak{m} = F_l/\mathfrak{m} \leq [f]_{\beta_l}/\mathfrak{m}$, then $h(b_l) \leq f(b_l)$ by Proposition 2.3.35 (ii), therefore $b_l \in \{h \leq f\} \subseteq D_1$, and thus (iii) (b) holds. Items (iii) (c) and (iii) (d) hold using the same arguments as in the proof of (I) $\Rightarrow$ (II) (iii) in Lemma 3.3.10. Define $D_1' := \{h \geq f\}$. Then $D_1' \cup D_1 \supseteq \{h \geq f\} \cup \{h \leq f\} = X$ and $D_1' \cap D_2 \subseteq \{h \geq f\} \cap \{h \leq g\} \subseteq \{f \leq g\}$, therefore (iv) (a) holds. Item (iv) (b) holds by the same arguments used to show that items (iii) (c) and (iii) (d) hold.

(II) $\Rightarrow$ (I). For each $l \in [n]$, pick $h_l \in C_{\text{s.a.}}(X)$ such that $F_l = [h_l]_{\beta_l}$. By item (i) and Lemma 3.3.9 there exist curve intervals $C_1', \ldots, C_n'$ of $\beta_1, \ldots, \beta_n$ (respectively) and $h' \in C_{\text{s.a.}}(X)$ such that $C_l' \subseteq \{h' = h_l\}$ for all $l \in [n]$. Using the same argument as in the proof of Lemma 3.3.10, it now suffices to prove that there exist curve intervals $C_1, \ldots, C_n$ of $\beta_1, \ldots, \beta_n$ (respectively) and $h \in C_{\text{s.a.}}(X)$ such that

$$\{h \leq f\} \subseteq D_1 \curlywedge D_2 \subseteq \{h \leq g\} \curlywedge \bigcurlywedge_{l \in [n]} C_l \subseteq \{h' = h\},$$

that is, such that

$$\left(\bigcurlywedge_{l \in [n]} C_l \subseteq \{h \geq h'\}\right) \curlywedge \{h \leq f\} \subseteq D_1 \curlywedge \left(D_2 \subseteq \{h \leq g\} \curlywedge \bigcurlywedge_{l \in [n]} C_l \subseteq \{h \leq h'\})\right).$$

By Lemma 2.4.18, it suffices in turn to show that there exist curve intervals $C_1, \ldots, C_n$ of $\beta_1, \ldots, \beta_n$ such that all the conditions below hold:

1. For all $l \in [n]$ there exists $h \in C_{\text{s.a.}}(X)$ such that

$$C_l \subseteq \{h \geq h'\} \text{ and } D_2 \subseteq \{h \leq g\}.$$

2. For all $l_1, l_2 \in [n]$ there exists $h \in C_{\text{s.a.}}(X)$ such that $C_{l_1} \subseteq \{h \geq h'\}$ and $C_{l_2} \subseteq \{h \leq h'\}$.

3. There exists $h \in C_{\text{s.a.}}(X)$ such that $\{h \leq f\} \subseteq D_1$ and $D_2 \subseteq \{h \leq g\}$.

4. For all $l \in [n]$ there exists $h \in C_{\text{s.a.}}(X)$ such that

$$\{h \leq f\} \subseteq D_1 \quad \text{and} \quad C_l \subseteq \{h \leq h'\}.$$

Item 2 holds trivially for any choice of curve intervals by taking $h := h'$, and item 3 holds by Proposition 2.4.25 (II) appealing to items (ii) and (iv) (a) in the statement of the lemma. It therefore remains to show that there exist curve intervals $C_1, \ldots, C_n$ of $\beta_1, \ldots, \beta_n$ such that conditions 1 and 4 above hold. In turn, this is equivalent to showing that for all $l \in [n]$ there exists a curve interval C_l of β_l such that the following two conditions hold for C_l:

1'. There exists $h \in C_{\text{s.a.}}(X)$ such that $C_l \subseteq \{h \geq h'\}$ and $D_2 \subseteq \{h \leq g\}$.

4'. There exists $h \in C_{\text{s.a.}}(X)$ such that $\{h \leq f\} \subseteq D_1$ and $C_l \subseteq \{h \leq h'\}$.

By items (I) and (II) of Proposition 2.4.25 it suffices to prove that for all $l \in [n]$ there exists a curve interval C_l of β_l such that

$$C_l \cap D_2 \subseteq \{h' \leq g\},\ C_l \cap \{h' \leq f\} \subseteq D_1, \text{ and } D_1' \cap C_l \subseteq \{f \leq h'\}. \qquad (*)$$

For what follows, recall that $C_l' \subseteq \{h' = h_l\}$ for all $l \in [n]$, so that in particular $[h']_{\beta_l} = [h_l]_{\beta_l} = F_l$. Fix $l \in [n]$; then exactly one of the following situations holds:

- $\{h' \leq f\} \in \mathfrak{f}_{\beta_l}$. In this case $F_l \leq [f]_{\beta_l}$, therefore by item (iii) (a) there exists a curve interval I of β_l such that $I \subseteq \{h' \leq f\} \cap D_1$; in particular $I \cap \{h' \leq f\} \subseteq D_1$.

- $\{h' \leq f\} \notin \mathfrak{f}_{\beta_l}$ and $b_l \in \{h' \leq f\}$. In this case there exists a curve interval I of β_l such that $I \cap \{h' \leq f\} = \{b_l\}$. Also, $F_l/\mathfrak{m} \leq [f]_{\beta_l}/\mathfrak{m}$, therefore $b_l \in D_1$ by item (iii) (b); in particular $I \cap \{h' \leq f\} \subseteq D_1$.

- $b_l \notin \{h' \leq f\}$. In this case exists a curve interval I of β_l such that $I \cap \{h' \leq f\} = \varnothing$; in particular $I \cap \{h' \leq f\} \subseteq D_1$.

In either of the cases above there exists a curve interval I of β_l such that $I \cap \{h' \leq f\} \subseteq D_1$. Using analogous arguments, there exists a curve interval I' of β_l such that $D_1' \cap I' \subseteq \{f \leq h'\}$ appealing to items (a') and (b'), and there exists a curve interval J of β_l such that $J \cap D_2 \subseteq \{h' \leq g\}$ appealing to items (iii) (c) and (iii) (d); see also the end of the proof of (II) $\Rightarrow$ (I) of Lemma 3.3.10. Since $I' \cap I \cap J \in \beta_l$, there exists a curve interval C_l of β_l such that $C_l \subseteq I' \cap I \cap J$; such curve interval satisfies $(*)$, and this concludes the proof. □

Lemma 3.3.12. *Let $f, g \in C_{\text{s.a.}}(X)$, $D_1, D_2 \in L_X$, $F_1, \ldots, F_n \in \mathcal{O}_R$, and $\beta_1, \ldots, \beta_n$ be pairwise distinct half-branches of X at $b_1, \ldots, b_n$. The following are equivalent:*

(I) *There exists $h \in C_{\text{s.a.}}(X)$ such that*

$$D_1 \subseteq \{h \geq f\} \curlywedge \{h \geq g\} \subseteq D_2 \curlywedge \bigcurlywedge_{l=1}^{n} F_l = [h]_{\beta_l}.$$

(II) *The following conditions hold:*

  (i) *For all $l_1, l_2 \in [n]$, if $b_{l_1} = b_{l_2}$, then $F_{l_1} - F_{l_2} \in \mathfrak{m}$.*

  (ii) *$D_1 \cap \{g \leq f\} \subseteq D_2$.*

  (iii) *For all $l \in [n]$, the following conditions hold:*

    (a) *If $F_l \geq [g]_{\beta_l}$, then $D_2 \in \mathfrak{f}_{\beta_l}$.*

    (b) *If $F_l/\mathfrak{m} \geq [g]_{\beta_l}/\mathfrak{m}$, then $b_l \in D_2$.*

    (c) *If $D_1 \in \mathfrak{f}_{\beta_l}$, then $F_l \geq [f]_{\beta_l}$.*

    (d) *If $D_1 \notin \mathfrak{f}_{\beta_l}$ and $b_l \in D_1$, then $F_l/\mathfrak{m} \geq [f]_{\beta_l}/\mathfrak{m}$.*

  (iv) *There exists $D_2' \in L_X$ such that the following conditions hold:*

    (a) *$D_2' \cup D_2 = X$ and $D_2' \cap D_1 \subseteq \{f \leq g\}$.*

    (b) *For all $l \in [n]$, the following conditions hold:*

      (a') *If $D_2' \in \mathfrak{f}_{\beta_l}$, then $F_l \leq [g]_{\beta_l}$.*

      (b') *If $D_2' \notin \mathfrak{f}_{\beta_l}$ and $b_l \in D_2'$, then $F_l/\mathfrak{m} \leq [g]_{\beta_l}/\mathfrak{m}$.*

Proof. Item (I) is equivalent to the statement that there exists $h \in C_{\text{s.a.}}(X)$ such that

$$\{h \leq -g\} \subseteq D_2 \curlywedge D_1 \subseteq \{h \leq -f\} \curlywedge \bigcurlywedge_{l=1}^{n} -F_l = [h]_{\beta_l},$$

and this statement is easily seen to be equivalent to item (II) by Lemma 3.3.11. □

Lemma 3.3.13. *Let $f, g \in C_{\text{s.a.}}(X)$, $D_1, D_2 \in L_X$, $F_1, \ldots, F_n \in \mathcal{O}_R$, and $\beta_1, \ldots, \beta_n$ be pairwise distinct half-branches of X at $b_1, \ldots, b_n$. The following are equivalent:*

(I) *There exists $h \in C_{\text{s.a.}}(X)$ such that*

$$\{h \leq f\} \subseteq D_1 \curlywedge \{h \geq g\} \subseteq D_2 \curlywedge \bigcurlywedge_{l=1}^{n} F_l = [h]_{\beta_l}.$$

(II) *The following conditions hold:*

(i) *For all $l_1, l_2 \in [n]$, if $b_{l_1} = b_{l_2}$, then $F_{l_1} - F_{l_2} \in \mathfrak{m}$.*

(ii) *For all $l \in [n]$, the following conditions hold:*

(a) *If $F_l \leq [f]_{\beta_l}$, then $D_1 \in \mathfrak{f}_{\beta_l}$.*

(b) *If $F_l/\mathfrak{m} \leq [f]_{\beta_l}/\mathfrak{m}$, then $b_l \in D_1$.*

(c) *If $F_l \geq [g]_{\beta_l}$, then $D_2 \in \mathfrak{f}_{\beta_l}$.*

(d) *If $F_l/\mathfrak{m} \geq [g]_{\beta_l}/\mathfrak{m}$, then $b_l \in D_2$.*

(iii) *There exist $D'_1, D'_2 \in L_X$ such that the following conditions hold:*

(a) *$D'_1 \cup D_1 = D'_2 \cup D_2 = X$.*

(b) *$D'_1 \cap \{g \leq f\} \subseteq D_2$ and $D'_2 \cap \{g \leq f\} \subseteq D_1$.*

(c) *$D'_1 \cap D'_2 \subseteq \{f \leq g\}$.*

(d) *For all $l \in [n]$, the following conditions hold:*

(a') *If $D'_1 \in \mathfrak{f}_{\beta_l}$, then $F_l \geq [f]_{\beta_l}$.*

(b') *If $D'_1 \notin \mathfrak{f}_{\beta_l}$ and $b_l \in D'_1$, then $F_l/\mathfrak{m} \geq [f]_{\beta_l}/\mathfrak{m}$.*

(c') *If $D'_2 \in \mathfrak{f}_{\beta_l}$, then $F_l \leq [g]_{\beta_l}$.*

(d') *If $D'_2 \notin \mathfrak{f}_{\beta_l}$ and $b_l \in D'_2$, then $F_l/\mathfrak{m} \leq [g]_{\beta_l}/\mathfrak{m}$.*

Proof. (I) $\Rightarrow$ (II). Item (i) holds using the same argument as in the proof of (I) $\Rightarrow$ (II) (i) in Lemma 3.3.10, and item (ii) holds using the same argument as in the proof of (I) $\Rightarrow$ [(II) (iii) (a) & (II) (iii) (b)] in Lemma 3.3.11. Define $D'_1 := \{h \geq f\}$ and $D'_2 := \{h \leq g\}$; the proof that item (iii) holds for this choice of $D'_1, D'_2 \in L_x$ is straightforward using the same arguments as in the proofs of the aforementioned lemmas.

(II) $\Rightarrow$ (I). For each $l \in [n]$, pick $h_l \in C_{\text{s.a.}}(X)$ such that $F_l = [h_l]_{\beta_l}$. By item (i) and Lemma 3.3.9 there exist curve intervals $C'_1, \ldots, C'_n$ of $\beta_1, \ldots, \beta_n$ (respectively) and

$h' \in C_{\text{s.a.}}(X)$ such that $C'_l \subseteq \{h' = h_l\}$ for all $l \in [n]$. Using the same argument as in the proof of Lemma 3.3.10, it now suffices to prove that there exist curve intervals $C_1, \ldots, C_n$ of $\beta_1, \ldots, \beta_n$ (respectively) and $h \in C_{\text{s.a.}}(X)$ such that

$$\{h \leq f\} \subseteq D_1 \curlywedge \{h \geq g\} \subseteq D_2 \curlywedge \bigcurlywedge_{l \in [n]} C_l \subseteq \{h' = h\},$$

that is, such that

$$\left(\bigcurlywedge_{l \in [n]} C_l \subseteq \{h \geq h'\} \right) \curlywedge \{h \leq f\} \subseteq D_1 \curlywedge$$
$$\left(\bigcurlywedge_{l \in [n]} C_l \subseteq \{h \leq h'\} \right) \curlywedge \{h \geq g\} \subseteq D_2.$$

By Lemma 2.4.18, it suffices in turn to show that there exist curve intervals $C_1, \ldots, C_n$ of $\beta_1, \ldots, \beta_n$ such that all the conditions below hold:

1. For all $l_1, l_2 \in [n]$ there exists $h \in C_{\text{s.a.}}(X)$ such that $C_{l_1} \subseteq \{h \geq h'\}$ and $C_{l_2} \subseteq \{h \leq h'\}$.

2. For all $l \in [n]$ there exists $h \in C_{\text{s.a.}}(X)$ such that
$$C_l \subseteq \{h \geq h'\} \text{ and } \{h \geq g\} \subseteq D_2.$$

3. For all $l \in [n]$ there exists there exists $h \in C_{\text{s.a.}}(X)$ such that
$$\{h \leq f\} \subseteq D_1 \text{ and } C_l \subseteq \{h \leq h'\}.$$

4. There exists $h \in C_{\text{s.a.}}(X)$ such that $\{h \leq f\} \subseteq D_1$ and $\{h \geq g\} \subseteq D_2$.

Item 1 holds for any choice of curve intervals by taking $h := h'$, and item 4 holds by Proposition 2.4.25 (IV) appealing to items (iii) (a) - (iii) (c). It therefore remains to show that there exist curve intervals $C_1, \ldots, C_n$ of $\beta_1, \ldots, \beta_n$ such that conditions 2 and 3 above hold. In turn, this is equivalent to showing that for all $l \in [n]$ there exists a curve interval C_l of β_l such that the following two conditions hold:

2'. There exists $h \in C_{\text{s.a.}}(X)$ such that $C_l \subseteq \{h \geq h'\}$ and $\{h \geq g\} \subseteq D_2$.

3'. There exists $h \in C_{\text{s.a.}}(X)$ such that $\{h \leq f\} \subseteq D_1$ and $C_l \subseteq \{h \leq h'\}$.

By items (II) and (III) of Proposition 2.4.25 it suffices to prove that for all $l \in [n]$ there exists a curve interval C_l of β_l such that

$$\begin{aligned} &C_l \cap \{g \leq h'\} \subseteq D_2, C_l \cap D'_2 \subseteq \{h' \leq g\}, \\ &C_l \cap \{h' \leq f\} \subseteq D_1, \text{ and } D'_1 \cap C_l \subseteq \{f \leq h'\}. \end{aligned} \tag{$*$}$$

Using the same arguments as at the end of the proofs of (II) $\Rightarrow$ (I) in Lemmas 3.3.10 and 3.3.11, it follows that:

- Items (ii) (a) and (ii) (b) imply that there exists a curve interval I of β_l such that $I \cap \{h' \leq f\} \subseteq D_1$.
- Items (ii) (c) and (ii) (d) imply that there exists a curve interval J of β_l such that $J \cap \{g \leq h'\} \subseteq D_2$.
- Items (a') and (b') imply that exists a curve interval I' of β_l such that $D'_1 \cap I' \subseteq \{f \leq h'\}$.
- Items (c') and (d') imply that exists a curve interval J' of β_l such that $J' \cap D'_2 \subseteq \{h' \leq g\}$.

Since $I' \cap J' \cap I \cap J \in \beta_l$, there exists a curve interval C_l of β_l such that $C_l \subseteq I' \cap J' \cap I \cap J$; such curve interval satisfies $(*)$, and this concludes the proof. □

Lemma 3.3.14. *Consider the $\mathscr{L}_X$-formula*

$$\begin{aligned} \exists x \Bigg(&\bigwedge_{i \in I_1} \xi_{1i} \sqsubseteq \{x \geq t_{1i}(\overline{z})\} \wedge \bigwedge_{i \in I_2} \{x \leq t_{2i}(\overline{z})\} \sqsubseteq \xi_{2i} \wedge \\ &\bigwedge_{i \in I_3} \xi_{3i} \sqsubseteq \{x \leq t_{3i}(\overline{z})\} \wedge \bigwedge_{i \in I_4} \{x \geq t_{4i}(\overline{z})\} \sqsubseteq \xi_{4i} \wedge \\ &\bigwedge_{l=1}^{n} Y_l = [x]_{\beta_l} \Bigg), \end{aligned} \tag{♣}$$

where

(i) *x is a home variable;*

(ii) *$I_1, I_2, I_3,$ and I_4 are disjoint finite index sets;*

(iii) *all ξ_{ki} are space variables, and all Y_l are germ variables;*

(iv) *all* $t_{ki}(\overline{z})$ *are* $\mathscr{L}^{C_{\text{s.a.}}(X)\text{-mod}}$*-terms; and*

(v) *all* β_l *are pairwise distinct half-branches of* X*.*

Then (♣) *is equivalent modulo* $\mathscr{M}_X$ *to the conjunction of all the following formulas:*

(a) *For all* $i \in I_1$ *and all* $j \in I_3$ *the formula*

$$\exists x \left(\xi_{1i} \sqsubseteq \{x \geq t_{1i}(\overline{z})\} \mathbin{\wedge\!\!\!\wedge} \xi_{3j} \sqsubseteq \{x \leq t_{3j}(\overline{z})\} \mathbin{\wedge\!\!\!\wedge} \bigwedge\!\!\!\bigwedge_{l=1}^{n} Y_l = [x]_{\beta_l} \right).$$

(b) *For all* $i \in I_2$ *and all* $j \in I_3$ *the formula*

$$\exists x \left(\{x \leq t_{2i}(\overline{z})\} \sqsubseteq \xi_{2i} \mathbin{\wedge\!\!\!\wedge} \xi_{3j} \sqsubseteq \{x \leq t_{3j}(\overline{z})\} \mathbin{\wedge\!\!\!\wedge} \bigwedge\!\!\!\bigwedge_{l=1}^{n} Y_l = [x]_{\beta_l} \right).$$

(c) *For all* $i \in I_1$ *and all* $j \in I_4$ *the formula*

$$\exists x \left(\xi_{1i} \sqsubseteq \{x \geq t_{1i}(\overline{z})\} \mathbin{\wedge\!\!\!\wedge} \{x \geq t_{4j}(\overline{z})\} \sqsubseteq \xi_{4j} \mathbin{\wedge\!\!\!\wedge} \bigwedge\!\!\!\bigwedge_{l=1}^{n} Y_l = [x]_{\beta_l} \right).$$

(d) *For all* $i \in I_2$ *and all* $j \in I_4$ *the formula*

$$\exists x \left(\{x \leq t_{2i}(\overline{z})\} \sqsubseteq \xi_{2i} \mathbin{\wedge\!\!\!\wedge} \{x \geq t_{4j}(\overline{z})\} \sqsubseteq \xi_{4j} \mathbin{\wedge\!\!\!\wedge} \bigwedge\!\!\!\bigwedge_{l=1}^{n} Y_l = [x]_{\beta_l} \right).$$

Proof. The proof is analogous to that of Lemma 2.4.18; in fact, the statement follows from Lemma 2.4.18. Clearly (♣) implies each of the formulas in items (a) - (d). Conversely, if $h_{ij} \in C_{\text{s.a.}}(X)$ $(i \in I_1 \dot\cup I_2,\ j \in I_3 \dot\cup I_4)$ witness each of the existential quantifiers of the formulas (a) - (d) then $h := \bigvee_{i \in I_1 \dot\cup\ I_2} \bigwedge_{i \in I_3 \dot\cup I_4} h_{ij}$ witnesses the existential quantifier of the formula (♣) by the moreover part of Lemma 2.4.18 and since

$$[h]_{\beta_l} = \left[\bigvee_{i \in I_1 \dot\cup I_2} \bigwedge_{i \in I_3 \dot\cup I_4} h_{ij} \right]_{\beta_l} = \max_{i \in I_1 \dot\cup I_2} \min_{i \in I_3 \dot\cup I_4} [h_{ij}]_{\beta_l} = Y_l$$

by Theorem 2.3.2 (III). □

Proposition 3.3.15. *Let* $n \in \mathbb{N}$ *and* $\beta_1, \ldots, \beta_n$ *be pairwise distinct half-branches of* X*. Every* $\mathscr{L}_X$*-formula of the form*

$$\exists x \left(\bigwedge\!\!\!\bigwedge_{i=1}^{m_1} \xi_{1i} = \{x \geq t_{1i}(\overline{z})\} \mathbin{\wedge\!\!\!\wedge} \bigwedge\!\!\!\bigwedge_{i=1}^{m_2} \xi_{2i} = \{x \leq t_{1i}(\overline{z})\} \mathbin{\wedge\!\!\!\wedge} \bigwedge\!\!\!\bigwedge_{l=1}^{n} Y_l = [x]_{\beta_l} \right) \qquad (\bigstar_2)$$

is equivalent modulo $\mathscr{M}_X^{\text{const}}$ *to an* $\mathscr{L}_X^{\text{const}}$*-formula without home quantifiers.*

Proof. Writing $\xi_{1i} = \{x \geq t_{1i}(\overline{z})\}$ as $\xi_{1i} \sqsubseteq \{x \geq t_{1i}(\overline{z})\} \curlywedge \{x \geq t_{1i}(\overline{z})\} \sqsubseteq \xi_{1i}$ and similarly for $\xi_{1i} = \{x \leq t_{1i}(\overline{z})\}$, it is clear that formulas of the form ($\bigstar_2$) are equivalent to formulas of the form (♣), therefore by Lemma 3.3.14 the formula ($\bigstar_2$) is equivalent to a conjunction of formulas of the following form:

(a) $\exists x\,(\xi_{1i} \sqsubseteq \{x \geq t_{1i}(\overline{z})\} \curlywedge \xi_{2j} \sqsubseteq \{x \leq t_{2j}(\overline{z})\} \curlywedge \bigcurlywedge_{l=1}^{n} Y_l = [x]_{\beta_l})$.

(b) $\exists x\,(\{x \leq t_{2i}(\overline{z})\} \sqsubseteq \xi_{2i} \curlywedge \xi_{2j} \sqsubseteq \{x \leq t_{2j}(\overline{z})\} \curlywedge \bigcurlywedge_{l=1}^{n} Y_l = [x]_{\beta_l})$.

(c) $\exists x\,(\xi_{1i} \sqsubseteq \{x \geq t_{1i}(\overline{z})\} \curlywedge \{x \geq t_{1j}(\overline{z})\} \sqsubseteq \xi_{1j} \curlywedge \bigcurlywedge_{l=1}^{n} Y_l = [x]_{\beta_l})$.

(d) $\exists x\,(\{x \leq t_{2i}(\overline{z})\} \sqsubseteq \xi_{2i} \curlywedge \{x \geq t_{1j}(\overline{z})\} \sqsubseteq \xi_{1j} \curlywedge \bigcurlywedge_{l=1}^{n} Y_l = [x]_{\beta_l})$.

It is now claimed that Lemmas 3.3.10, 3.3.11, 3.3.12, and 3.3.13 respectively imply that each of the formulas (a), (b), (c), and (d) above are equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers, from which the proof concludes. To this end it suffices to argue that each of the conditions within item (II) of each of the aforementioned lemmas can be expressed as $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers. But this is clear recalling that the formula $\texttt{PrimF}_{\beta_l}(\xi)$ defines the prime filter $\mathfrak{f}_{\beta_l}$ in L_X (see Lemma 3.3.6), and noting that

$$F_1/\mathfrak{m} \leq F_2/\mathfrak{m} \iff \mathcal{O}_R \models \exists Z(Z \geq 0 \curlywedge \mathfrak{m}((F_2 - F_1) - Z))$$

by definition of the total order on $\mathcal{O}_R/\mathfrak{m}$ (see [KS22, Remarks 2.2.6 (1)]). □

3.4 Proof of the main theorem (Theorem 3.1.8)

By Lemma 3.2.6 and Lemma 3.2.9 it suffices to show that every $\mathscr{L}_X$-formula of the form

$$\exists x \left[\bigcurlywedge_{i=1}^{m} \xi_i = \{f_i \cdot x \geq t_i(\overline{z})\} \curlywedge \bigcurlywedge_{j=1}^{n} Y_j = [x]_{\beta_j} \right] \qquad (♣_2)$$

is equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers, where

(i) x is a home variable;

(ii) $\xi_1, \ldots, \xi_m$ are space variables and $Y_1, \ldots, Y_n$ are germ variables;

(iii) $t_1(\overline{z}), \ldots, t_m(\overline{z})$ are $\mathscr{L}^{C_{\text{s.a.}}(X)\text{-mod}}$-terms;

(iv) $f_i \in C_{\text{s.a.}}(X)$ are scalar functions such that $f_i \geq 0$ or $f_i \leq 0$ for all $i \in [m]$; and

(v) $\beta_1, \ldots, \beta_n$ are pairwise distinct half-branches of X.

In what follows, every occurrence of "equivalent" means "equivalent modulo $\mathscr{M}_X^{\text{const}}$". Formulas of the form (✤_2) are clearly equivalent to formulas of the form

$$\exists x \left[\bigwedge\!\!\!\bigwedge_{i=1}^{d} \xi_i = \{f_i \cdot x \geq t_i(\overline{z})\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{i=d+1}^{m} \xi_i = \{f_i \cdot x \leq t_i(\overline{z})\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{j=1}^{n} Y_j = [x]_{\beta_j} \right], \qquad (\text{✤}_3)$$

where $f_i \geq 0$ for all $i \in [m]$. In turn, since $\{g_1 \leq g_2\} = \{f \cdot g_1 \leq f \cdot g_2\}$ and for all $g_1, g_2, f \in C_{\text{s.a.}}(X)$ with $f \geq 0$, by multiplying each inequality $f_i \cdot x \geq t_i(\overline{z})$ and $f_i \cdot x \leq t_i(\overline{z})$ in (✤_3) by a suitable scalar, it might be assumed that $f_1 = \cdots = f_m =: f \geq 0$. Therefore it remains to eliminate home quantifiers in formulas of the form

$$\exists x \left[\bigwedge\!\!\!\bigwedge_{i=1}^{d} \xi_i = \{f \cdot x \geq t_i(\overline{z})\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{i=d+1}^{m} \xi_i = \{f \cdot x \leq t_i(\overline{z})\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{j=1}^{n} Y_j = [x]_{\beta_j} \right], \qquad (\text{✤}_4)$$

where all symbols are as above, and where $f \in C_{\text{s.a.}}(X)$ is a fixed non-negative scalar. The formula (✤_4) is equivalent to

$$\exists y \exists x \left[\bigwedge\!\!\!\bigwedge_{i=1}^{d} \xi_i = \{y \geq t_i(\overline{z})\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{i=d+1}^{m} \xi_i = \{y \leq t_i(\overline{z})\} \wedge\!\!\!\wedge y = f \cdot x \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{j=1}^{n} Y_j = [x]_{\beta_j} \right]$$

that is, equivalent to

$$\begin{aligned} \exists y \Bigg[& \bigwedge\!\!\!\bigwedge_{i=1}^{d} \xi_i = \{y \geq t_i(\overline{z})\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{i=d+1}^{m} \xi_i = \{y \leq t_i(\overline{z})\} \wedge\!\!\!\wedge \\ & \underbrace{\exists x \left(y = f \cdot x \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{j=1}^{n} Y_j = [x]_{\beta_j} \right) \Bigg]}_{(\dagger)}. \end{aligned} \qquad (\text{✤}_5)$$

Applying Corollary 3.3.8 to the subformula ($\dagger$), the formula (✤_5) is equivalent to one of the form

$$\begin{aligned} \exists y \Bigg[& \bigwedge\!\!\!\bigwedge_{i=1}^{d} \xi_i = \{y \geq t_i(\overline{z})\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{i=d+1}^{m} \xi_i = \{y \leq t_i(\overline{z})\} \wedge\!\!\!\wedge \\ & \exists \zeta \exists \overline{Y'} \left(\psi(\zeta, \overline{Y}, \overline{Y'}) \wedge\!\!\!\wedge \zeta = \{y = 0\} \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{l=1}^{n'} Y'_l = [y]_{\beta'_l} \right), \end{aligned} \qquad (\text{✤}_6)$$

where $\psi(\zeta, \overline{Y}, \overline{Y'})$ is a Boolean combination of space formulas with parameters and germ formulas with parameters (hence a formula without home quantifiers), and

$\beta'_1, \ldots, \beta'_{n'}$ are pairwise distinct half-branches. Since $\psi(\zeta, \overline{Y}, \overline{Y'})$ does not contain the variable y, the formula (✽$_6$) is in turn equivalent to

$$\exists\zeta\exists\overline{Y'}\Bigg[\psi(\zeta,\overline{Y},\overline{Y'}) \curlywedge \exists y\Bigg(\bigcurlywedge_{i=1}^{d}\xi_i = \{y \geq t_i(\overline{z})\} \curlywedge \bigcurlywedge_{i=d+1}^{m}\xi_i = \{y \leq t_i(\overline{z})\} \curlywedge \zeta = \{y = 0\} \curlywedge \bigcurlywedge_{l=1}^{n'} Y'_l = [y]_{\beta'_l}\Bigg)\Bigg]. \tag{✽$_7$}$$

Let ζ^+ and ζ^- be new space variables. Then (✽$_7$) is equivalent to

$$\exists\zeta\exists\overline{Y'}\exists\zeta^+\zeta^-\Bigg[\psi(\zeta,\overline{Y},\overline{Y'}) \curlywedge \zeta = \zeta^+ \sqcap \zeta^-$$

$$(\dagger\dagger)\begin{cases}\exists y\Bigg(\bigcurlywedge_{i=1}^{d}\xi_i = \{y \geq t_i(\overline{z})\} \curlywedge \zeta^+ = \{y \geq 0\} \curlywedge \\ \bigcurlywedge_{i=d+1}^{m}\xi_i = \{y \leq t_i(\overline{z})\} \curlywedge \zeta^- = \{y \leq 0\} \curlywedge \\ \bigcurlywedge_{l=1}^{n'} Y'_l = [y]_{\beta'_l}\Bigg)\Bigg].\end{cases}$$

and the subformula (††) is equivalent to an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers by Proposition 3.3.15, concluding thus the proof.

3.5 Decidability

This section is devoted to harvest the decidability result (Proposition 3.5.3) obtained from Theorem 3.1.8 under the additional hypothesis that R is a recursive real closed field (Example 2.1.21). All the key notions of this section (namely recursive language, recursive structure, and decidable structure) are defined and discussed in Subsection 2.1.2.

Lemma 3.5.1. *Let R be a recursive real closed field.*

(i) *The partially ordered ring $C_{\text{s.a.}}(R)$ is recursive, that is, it is a recursive $\mathscr{L}^{\text{poring}}$-structure, where $\mathscr{L}^{\text{poring}} := \mathscr{L}^{\text{ring}}(\leq)$.*

(ii) *The ring of germs $\mathcal{O}_R$ is a recursive $\mathscr{L}^{\text{ring}}(\leq, \mathtt{m})$-structure.*

(iii) *The lattice of zero sets L_R of is a recursive $\mathscr{L}^{\text{lat}}(\top, \bot)$-structure.*

Proof. By Proposition 2.1.23, there exists a Gödel numbering $\ulcorner - \urcorner$ of $\mathscr{L}^{\text{poring}}(R)$ such that R is decidable, that is, such that the $\mathscr{L}^{\text{poring}}(R)$-theory of (R, R) is decidable.

(i). Define $\alpha : C_{\text{s.a.}}(R) \hookrightarrow \omega$ by

$$\alpha(f) := \min\{n \in \omega \mid n = \ulcorner \varphi \urcorner \text{ where } \varphi \in \mathscr{L}^{\text{poring}}(R)\text{-Fml defines } f\},$$

noting that α is a well-defined injective function.

Claim. $\alpha(C_{\text{s.a.}}(R)) \subseteq \omega$ is recursive.

Proof of Claim. Pick $n \in \omega$ arbitrarily. Since $\mathscr{L}^{\text{poring}}(R)$ is a recursive language, it can be decided if $n = \ulcorner \varphi(x, y) \urcorner$ for some $\varphi(x, y) \in \mathscr{L}^{\text{poring}}(R)$-Fml with exactly two free variables. Similarly, since R is decidable it can be decided if an $\mathscr{L}^{\text{poring}}(R)$-formula $\varphi(x, y)$ in two free variable defines the graph of a continuous function $R \longrightarrow R$, therefore the set

$$S := \{n \in \omega \mid n = \ulcorner \varphi \urcorner \text{ where } \varphi \in \mathscr{L}^{\text{poring}}(R)\text{-Fml} \\ \text{defines the graph of some } f \in C_{\text{s.a.}}(R)\}$$

is a recursive subset of ω. Using again the fact that R is decidable, it follows that

$$T := \{(n_1, n_2) \in S \times S \mid n_1 = \ulcorner \varphi_1 \urcorner,\ n_2 = \ulcorner \varphi_2 \urcorner, \text{ and } \varphi_1, \varphi_2 \in \mathscr{L}^{\text{poring}}(R)\text{-Fml} \\ \text{define the graph of the same } f \in C_{\text{s.a.}}(R)\}$$

is recursive, and the map $\mu : S \longrightarrow S$ given by $\mu(n) := \min\{n' \in S \mid (n, n') \in T\}$ is recursive with recursive image ($\mu(S)$ is exactly the set of fixed points of μ). For each $f \in C_{\text{s.a.}}(R)$ let φ_f be any $\mathscr{L}^{\text{poring}}(R)$-formula defining the graph of f; then $\alpha(f) = \mu(\ulcorner \varphi_f \urcorner)$, therefore $\alpha(C_{\text{s.a.}}(R)) = \mu(S)$ and thus the claim follows. $\square_{\text{Claim}}$.

If φ_f and φ_g are $\mathscr{L}^{\text{poring}}(R)$-formulas defining f and g (respectively), write φ_{f+g} for the formula $\exists y_1, y_2 (z = y_1 + y_2 \curlywedge \varphi_f(x, y_1) \curlywedge \varphi_g(x, y_2))$, noting that this defines the recursive map $S^2 \longrightarrow S$ given by $(\ulcorner \varphi_f \urcorner, \ulcorner \varphi_g \urcorner) \mapsto \ulcorner \varphi_{f+g} \urcorner$, where S is as in the proof of the claim above. Define $\alpha(f) + \alpha(g) := \alpha(f + g)$ for all $f, g \in C_{\text{s.a.}}(R)$; then $\alpha(f) + \alpha(g) = \mu(\ulcorner \varphi_{f+g} \urcorner)$ where μ is as defined in the proof of the claim above, from which it follows that the defined addition is recursive on $\alpha(C_{\text{s.a.}}(R))$. Similar arguments show that the other symbols in $\mathscr{L}^{\text{poring}}$ have recursive interpretations on $\alpha(C_{\text{s.a.}}(R))$, from which it follows that $C_{\text{s.a.}}(R)$ is recursive, see Remark 2.1.20 (i).

(ii). Let $\mathfrak{p}_{0^+} := \{f \in C_{\text{s.a.}}(R) \mid \exists \varepsilon \in (0, 1) \text{ such that } f_{\restriction [0,\varepsilon]_R} = 0\}$ and $\mathfrak{m}_0 := \{f \in C_{\text{s.a.}}(R) \mid f(0) = 0\}$. Using the recursive presentation of $C_{\text{s.a.}}(R)$ given in item

(i) together with decidability of R it follows that $\mathfrak{p}_{0+}$ and $\mathfrak{m}_0$ are recursive ideals of $C_{\text{s.a.}}(R)$ (see Example 2.1.22), therefore $C_{\text{s.a.}}(R)/\mathfrak{p}_{0+} \cong \mathcal{O}_R$ is a recursive partially ordered ring with recursive ideal $\mathfrak{m}_0/\mathfrak{p}_{0+} \cong \mathfrak{m}$ (see Example 2.3.36).

(iii). Note first that a proof analogous to the one used in item (i) to show that addition is recursive on $\alpha(C_{\text{s.a.}}(R))$ shows that the lattice operations $\vee$ and $\wedge$ are recursive on $\alpha(C_{\text{s.a.}}(R))$; in other words, $C_{\text{s.a.}}(R)$ is a recursive lattice-ordered ring. As in item (ii), using the recursive presentation of $C_{\text{s.a.}}(R)$ it follows that the equivalence relation on $C_{\text{s.a.}}(R)$ given by $f \sim g$ if and only if $\{f \geq 0\} = \{g \geq 0\}$ is recursive. The quotient of the recursive structure $(C_{\text{s.a.}}(R), \leq, \vee, \wedge, 0, -1)$ by the recursive equivalence relation $\sim$ is exactly the $\mathscr{L}^{\text{lat}}(\top, \bot)$-structure L_R, therefore the latter structure also is recursive (see [Mon21, Lemma I.11]). □

Lemma 3.5.2. *Let R be a recursive real closed field.*

(i) *The ring of germs $\mathcal{O}_R$ is a decidable $\mathscr{L}^{\text{ring}}(\leq, \mathtt{m})$-structure.*

(ii) *The lattice of zero sets L_X is a decidable $\mathscr{L}^{\text{lat}}(\top, \bot)$-structure.*

Proof. (i). Combine Proposition 2.1.23, Proposition 2.3.13 (i), and Lemma 3.5.1 (ii).

(ii). The following proof is a refinement of the proof of Proposition 2.4.32 (ii). By [Tre16, 4.1. (vii) (a)] the lattice L_X is parametrically definable in the lattice L_R, therefore it suffices to show that L_R is a decidable $\mathscr{L}^{\text{lat}}(\top, \bot)$-structure. Let $W(R, \leq)$ be the weak monadic second-order structure of the linear order $(R, \leq)$, that is, $W(R, \leq)$ is the poset of finite subsets of R expanded by all 0-definable (in R) subsets of R^n ($n \in \mathbb{N}$), see [Tre17, Definition 2.1]. By [Tre17, Proposition 3.2. (i)] and its proof, the recursive lattice L_R (Lemma 3.5.1 (iii)) is effectively interpretable in $W(R, \leq)$, therefore it suffices in turn to show that $W(R, \leq)$ is a decidable structure. Let S2S be the infinite binary tree $2^{<\omega}$ with two successor functions $\sigma \mapsto \sigma^\frown 0$ and $\sigma \mapsto \sigma^\frown 1$ (see [Rab69]); by [Tre17, Remark 3.5 (iii)] and the fact that R is countable, $W(R, \leq)$ is isomorphic to a reduct of $W(\text{S2S})$ (the weak monadic second-order structure of S2S). By Theorem 1.1 and Corollary 1.9 in [Rab69] (see also Theorem 3.4 and Corollary 3.6 in [Tre17]), the theory of $W(\text{S2S})$ is decidable. Since every finite subset of S2S is 0-definable in S2S, every element of $W(\text{S2S})$ is 0-definable in $W(\text{S2S})$, from which it follows that $W(\text{S2S})$ is a decidable structure, and thus so is $W(R, \leq)$, as required. □

Proposition 3.5.3. *Let R be a recursive real closed field.*

(i) *The language $\mathscr{L}_X^{\text{const}}$ is recursive and every $\mathscr{L}_X^{\text{const}}$-formula is effectively equivalent modulo $\mathscr{M}_X^{\text{const}}$ to an $\mathscr{L}_X^{\text{const}}$-formula of the form (♠).*

(ii) *The structure $\mathscr{M}_X^{\text{const}}$ has a decidable $\mathscr{L}_X^{\text{const}}$-theory. In particular, the $\mathscr{L}_X$-theory of $\mathscr{M}_X$ is decidable, and the $\mathscr{L}^{\ell\text{-}C_{\text{s.a.}}(X)\text{-mod}}$-theory of the lattice-ordered module $C_{\text{s.a.}}(X)$ is decidable.*

Proof. (i). If $\varphi(\overline{z}, \overline{\zeta}, \overline{Z})$ is an $\mathscr{L}_X^{\text{const}}$-formula, then the proof of Theorem 3.1.8 shows how to explicitly construct an $\mathscr{L}_X^{\text{const}}$-formula without home quantifiers equivalent to $\varphi(\overline{z}, \overline{\zeta}, \overline{Z})$ modulo $\mathscr{M}_X^{\text{const}}$, therefore this equivalence is effective whenever $\mathscr{L}_X^{\text{const}}$ is a recursive language. To prove that $\mathscr{L}_X^{\text{const}}$ is recursive, note first that the language $(\mathscr{L}_X^{\text{const}})_{\restriction\Sigma}$ is exactly the disjoint union of the languages $\mathscr{L}^{\text{lat}}(L_X)$ and $\mathscr{L}^{\text{ring}}(\leq, \mathtt{m}, \mathcal{O}_R)$ (which are recursive by Remarks 2.1.11 and 2.1.12), therefore by choosing any recursive presentations of these two latter languages, a standard coding argument (such as using a recursive bijection $\omega^2 \longrightarrow \omega$ as in [Mur99, p. 39]) shows that $(\mathscr{L}_X^{\text{const}})_{\restriction\Sigma}$ is recursive. Again by standard coding, any recursive presentation of $(\mathscr{L}_X^{\text{const}})_{\restriction\Sigma}$ can be extended to a recursive presentation of $\mathscr{L}_X^{\text{const}}$: this is possible since all but finitely many non-logical symbols in $\mathscr{L}_X^{\text{const}} \setminus (\mathscr{L}_X^{\text{const}})_{\restriction\Sigma}$ are unary, therefore the condition on the arity maps being partial recursive in Definition 2.1.10 can always be satisfied.

(ii). Since $\mathscr{M}_X$ and $C_{\text{s.a.}}(X)$ are both reducts of $\mathscr{M}_X^{\text{const}}$, it suffices to show that $\mathscr{M}_X^{\text{const}}$ has a decidable $\mathscr{L}_X^{\text{const}}$-theory T, that is, that there exists a recursive presentation $\lceil - \rceil$ of $\mathscr{L}_X^{\text{const}}$ with corresponding Gödel numbering $\ulcorner - \urcorner$ such that $\ulcorner T \urcorner$ is recursive. Since L_X and $\mathcal{O}_R$ are both decidable by Lemma 3.5.2, the proof of (i) shows that there exists a recursive presentation of $\mathscr{L}_X^{\text{const}}$ with corresponding Gödel numbering $\ulcorner - \urcorner$ such that $\ulcorner T_1 \urcorner$ and $\ulcorner T_2 \urcorner$ are both recursive, where T_1 and T_2 are the elementary diagrams of L_X and $\mathcal{O}_R$, that is, the theories of L_X and $\mathcal{O}_R$ in the languages $\mathscr{L}_1$ and $\mathscr{L}_2$ obtained by restricting $\mathscr{L}_X^{\text{const}}$ to the space and germ sorts, respectively. Let φ be an $\mathscr{L}_X^{\text{const}}$-sentence. By item (i) above, one can effectively find a Boolean combination ψ of $\mathscr{L}_1$-sentences ψ_{1i} and $\mathscr{L}_2$-sentences ψ_{2j} such that $\ulcorner \varphi \urcorner \in \ulcorner T \urcorner$ if and only if $\ulcorner \psi \urcorner \in \ulcorner T \urcorner$; by choice of ψ, it follows that $\ulcorner \psi \urcorner \in \ulcorner T \urcorner$ if and only if the corresponding Boolean combination of conditions of the form $\ulcorner \psi_{1i} \urcorner \in \ulcorner T_1 \urcorner$ and $\ulcorner \psi_{2j} \urcorner \in \ulcorner T_2 \urcorner$ holds, and since $\ulcorner T_1 \urcorner$ and $\ulcorner T_2 \urcorner$ are recursive, it follows that $\ulcorner \varphi \urcorner \in \ulcorner T \urcorner$ can be decided, as required. □

3.6 Concluding remarks

The key geometric ingredients in the proof of Theorem 3.1.8 are o-minimality of the real closed field R, the fact that X is one-dimensional, and the semi-algebraic Tietze extension theorem. The Tietze extension theorem also holds for continuous definable functions in o-minimal expansions of real closed fields, see [Dri98, Chapter 8, Corollary 3.10]. In particular, Theorem 3.1.8 also holds for the three-sorted structure analogue of $\mathscr{M}_X^{\text{const}}$ obtained by replacing the home sort with the lattice-ordered module (over itself) of continuous definable functions $C_{\text{def}}(X)$ on a definable curve X over an o-minimal expansion $\overline{R}$ of a real closed field R, and by replacing the space and germ sorts by the corresponding lattice of zero sets and ring of germs at a half-branch, respectively. The ring of germs of $C_{\text{def}}(X)$ at a half-branch of X is a $\text{Th}(\overline{R})$-convex subring of $\overline{R}$ in the sense of [DL95], where $\text{Th}(\overline{R})$ is the theory of $\overline{R}$ in the language of $\overline{R}$, therefore its theory is model complete whenever $\text{Th}(\overline{R})$ is. In particular, the proof of decidability given in Section 3.5 shows that decidability in the continuous definable case also follows under the extra hypotheses that $\overline{R}$ is a decidable structure (hence also recursive) with model complete theory $\text{Th}(\overline{R})$.

Continuing on decidability issues, a statement stronger than that given in Proposition 3.5.3 (ii) would be that if R is a recursive real closed field, then $\mathscr{M}_X$ is a decidable $\mathscr{L}_X$-structure, that is, that the elementary diagram of $\mathscr{M}_X$ is a decidable theory. The proof of such a statement would require a careful recursive presentation of the language $\mathscr{L}_X$ expanded by constants for all elements of the three sorts in $\mathscr{L}_X$, as well as a proof that with such recursive presentation the zero set map $C_{\text{s.a.}}(X) \twoheadrightarrow L_X$ and the germ maps $[-]_\beta : C_{\text{s.a.}}(X) \twoheadrightarrow \mathcal{O}_R$ are recursive.

To conclude, it will be now pointed where the main problem arises in trying to carry the proof of Theorem 3.1.8 as outlined in Subsection 3.1.2 to the structure $\mathscr{M}_X^{\text{ring}}$ obtained by replacing the module scalar functions on the home sort of $\mathscr{M}_X$ (or $\mathscr{M}_X^{\text{const}}$) by the full (binary) multiplication, so that the home sort becomes an $\mathscr{L}^{\ell\text{-ring}}$-structure in $\mathscr{M}_X^{\text{ring}}$, where $\mathscr{L}^{\ell\text{-ring}} := \{+, -, \cdot, 0, 1, \leq, \vee, \wedge\}$.

Using [DM95, Fact F6] in place of Proposition 3.2.2, one can easily show that every formula without home quantifiers is equivalent modulo $\mathscr{M}_X^{\text{ring}}$ to a formula of the form (♠), where in this case the terms $t_i(\overline{z})$ and $s_j(\overline{z})$ are $\mathscr{L}^{\ell\text{-ring}}$-terms. The main issue comes in the elimination of home quantifiers in formulas of the form ($\bigstar_1$). As shown in Lemmas 3.3.1 and 3.3.2, expressing that $f \in C_{\text{s.a.}}(X)$ divides $g \in C_{\text{s.a.}}(X)$ in the germ sort depends on all the half-branches at the boundary points of $\{f = 0\}$. In particular, if f is not fixed (as it would be the case in $\mathscr{M}_X^{\text{ring}}$), then these half-branches vary both in number and in the points at which they are centred as f varies, and thus depending on what f is, the number of conjuncts of germ formulas in item (II) of Lemma 3.3.2 varies.

Chapter 4

Local Real Closed SV-Rings of Finite Rank

Recall the fix the following terminology and notation for this chapter:

(i) Let $\{A_i\}_{i \in I}$ be a non-empty set of rings.

  (i) For all $j \in I$, let $\pi_j : \prod_{i \in I} A_i \twoheadrightarrow A_j$ be the projection map.

  (ii) A ring A is a *subdirect product of* $\{A_i\}_{i \in I}$ if A is a subring of $\prod_{i \in I} A_i$ and $\pi_{i \restriction A} : A \longrightarrow A_i$ is surjective for all $i \in I$; if A is a subdirect product of $\{A_i\}_{i \in I}$, define $p_i := \pi_{i \restriction A}$ for all $i \in I$.

(ii) If A is a ring, then $\mathrm{Spec}^{\min}(A)$ is its set of minimal prime ideals.

(iii) If A and B are local rings with respective unique maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$, then an injective ring homomorphism $f : A \hookrightarrow B$ is *local* if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

4.1 Introduction

The present chapter studies n-fold fibre products of non-trivial real closed valuation rings along surjective ring homomorphisms onto a fixed domain, where $n \in \mathbb{N}^{\geq 2}$. This "bottom-up" definition of this class of rings has an equivalent "top-down" description, namely, these rings are exactly local real closed SV-rings of finite rank with one branching ideal.

Survaluation rings (SV-rings for short) were first introduced in [HW92] in connection to rings $C(X)$ of continuous real-valued functions on a completely regular topological space X; in the aforementioned paper, the authors call the ring $C(X)$ an SV-ring if $C(X)/\mathfrak{p}$ is a valuation ring for every prime ideal $\mathfrak{p}$ of $C(X)$, and X an SV-space if $C(X)$ is an SV-ring. A canonical partial order on the rings $C(X)$ is defined by setting $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$; this partial order is a lattice-order which gives $C(X)$ the structure of an f-ring ([GJ60], [BKW77]), and this motivated the study of SV-rings within the class of f-rings in [HL93] (see also [Hen+94], [Lar11], as well as the survey [Lar10]). SV-rings can also be studied without the presence of a partial order: Schwartz defines in [Sch10b] a commutative and unital ring A to be an SV-ring if $A/\mathfrak{p}$ is a valuation ring for all prime ideals $\mathfrak{p}$ of A, and this is what is meant here by an SV-ring (Definition 4.2.1).

The article [Sch10b] contains a systematic study of SV-rings and it is the main reference on SV-rings for the present work. [Sch10b] also opened up the door for the model-theoretic study of SV-rings by proving the first results on axiomatizability (in the sense of model theory) of SV-rings in the language of rings $\mathscr{L} := \{+, -, \cdot, 0, 1\}$. In particular, it is shown in [Sch10b, Section 3] that the question of whether a class of SV-rings is elementary or not is tightly connected with the *rank* of the rings in this class; the rank of a prime ideal $\mathfrak{p}$ in a ring A is defined as the number (which is either a natural number or ∞) of minimal prime ideals $\mathfrak{q}$ of A such that $\mathfrak{q} \subseteq \mathfrak{p}$, and the rank of the ring A is the supremum of the ranks of its prime ideals (Definition 4.2.6), therefore a local ring has finite rank if and only if it has finitely many minimal prime ideals.

The rings $C(X)$ are particular examples of real closed rings in the sense of Schwartz, see Section 2.3, as well as [Sch89], [Sch86], [Sch97], [SM99, Section 12], and [Tre07]. The terminology "real closed ring" was first coined by Cherlin and Dickmann in [CD86] and [CD83], and some results in the literature about real closed rings refer to real closed rings in the sense of Cherlin and Dickmann (e.g. [MMS00]); in this chapter a real closed ring is always meant to be a real closed ring in the sense of Schwartz. Real closed rings in the sense of Cherlin and Dickmann are exactly real closed rings which are also valuation rings, that is, they are real closed valuation rings, see Subsection 2.3.1 and [Sch09]). Equivalently, these are local real closed SV-rings of rank 1, see

Corollary 4.2.8.

Non-trivial real closed valuation rings (i.e., those which are not fields) are exactly proper convex subrings of real closed fields, and this close relationship between these two classes of rings entails that non-trivial real closed valuation rings have many of the good model-theoretic properties of real closed fields ([CD83], [Bec83]). In particular, the class of non-trivial real closed valuation rings is elementary in the language of rings $\mathscr{L}$, and its theory is complete, decidable, and NIP ([CD83], [MMS00]). Here NIP stands for "not independence property", and it is a combinatorial property of first-order theories that can be described as certain families of definable sets having finite VC-dimension, see [Sim15]. It follows that the class of local real closed SV-rings of rank 1 splits into the classes of models of two complete, decidable, and NIP $\mathscr{L}$-theories, namely, the $\mathscr{L}$-theory RCF of real closed fields and the $\mathscr{L}$-theory RCVR of non-trivial real closed valuation rings.

Local real closed SV-rings of rank $n \in \mathbb{N}^{\geq 2}$ are exactly those rings obtained by taking iterated fibre products of finitely many non-trivial real closed valuation rings along surjective ring homomorphisms onto domains, see Theorem 4.4.2 for a precise formulation of this statement. Moreover, the class of local real closed SV-rings of rank $n \in \mathbb{N}^{\geq 2}$ is elementary in the language $\mathscr{L}$; this follows from [Sch10b, Proposition 2.2 and Corollary 3.16], but an equivalent axiomatization T_n for this class of rings is given in Definition 4.5.3.

A very particular class of local real closed SV-rings of rank $n \in \mathbb{N}^{\geq 2}$ is the one whose rings have exactly one branching ideal (Definition 4.3.1 and Lemma 4.4.7): a prime ideal $\mathfrak{q}$ in a ring A is defined to be a branching ideal if there exist distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq A$ such that $\mathfrak{p}_1, \mathfrak{p}_2 \subsetneq \mathfrak{q}$ and $\mathfrak{q} = \mathfrak{p}_1 + \mathfrak{p}_2$. Local real closed rings of rank $n \in \mathbb{N}^{\geq 2}$ have at least one branching ideal and at most $n-1$ branching ideals (Remarks 4.3.2 (ii) and 4.3.7), so those with exactly one branching ideal are the simplest rings in this class; moreover, there exists an $\mathscr{L}$-sentence $\varphi_{\mathrm{br},n}$ (Definition 4.5.4) such that for all local real closed rings A of rank n, $A \models \varphi_{\mathrm{br},n}$ if and only if A has exactly one branching ideal (Lemma 4.5.5).

If A is a local real closed ring of rank n with unique maximal ideal $\mathfrak{m}_A$ and with a unique branching ideal $\mathfrak{b}_A$, then either $\mathfrak{b}_A = \mathfrak{m}_A$ or $\mathfrak{b}_A \subsetneq \mathfrak{m}_A$, and this is an elementary property of the ring A (Proposition 4.3.5). In particular, the elementary class of local real closed SV-rings of rank n with exactly one branching ideal splits into two elementary classes of rings, namely, local real closed SV-rings of rank n with exactly one branching ideal $\mathfrak{b}_A$ such that $\mathfrak{b}_A = \mathfrak{m}_A$ (called for brevity rings of *type* $(n, 1)$, see Definition 4.4.8), and local real closed SV-rings of rank n with exactly one branching ideal $\mathfrak{b}_A$ such that $\mathfrak{b}_A \subsetneq \mathfrak{m}_A$ (called for brevity rings of *type* $(n, 2)$); geometric examples of rings of type $(n, 1)$ are rings of germs of continuous semi-algebraic functions $X \longrightarrow R$ at a point $x \in X$, where $X \subseteq R^m$ is a semi-algebraic curve over a real closed field R (Example 4.4.9). The main goal of this chapter is to provide a first model-theoretic analysis of the theories $T_{n,1}$ and $T_{n,2}$ or rings of type $(n, 1)$ and of type $(n, 2)$, respectively. Amongst other things, it is shown in Section 4.5 that $T_{n,1}$ and $T_{n,2}$ are complete, decidable, and NIP.

Much of the work towards proving the model-theoretic results in Section 4.5 rests on having good algebraic descriptions of local real closed SV-rings of finite rank and of their branching ideals, and this is the content of Sections 4.3 and 4.4. In particular, Theorem 4.4.2 is a structure theorem for local real closed SV-rings of finite rank which is deduced from a structure theorem for reduced local SV-rings rings of finite rank (Theorem 4.2.22), and Proposition 4.3.5 gives various equivalent conditions for the maximal ideal in a local real closed ring of finite rank to be a branching ideal; Proposition 4.3.5 then yields several equivalent characterizations of branching ideals in rings of this latter class (Remark 4.3.6).

4.1.1 Structure of the chapter

Section 4.2 starts by collecting the relevant material on SV-rings and on ranks of rings; in particular, Lemma 4.2.9 (III) and Corollary 4.2.11 describe minimal prime ideals in reduced local rings of finite rank as annihilator ideals of elements, and this is crucially used in the model-theoretic analysis of local real closed SV-rings of finite rank. The remaining part of Section 4.2 is devoted to prove a structure theorem for reduced local SV-rings of finite rank (Theorem 4.2.22), where the key algebraic result being used in the proof is Goursat's lemma for rings (Lemma 4.2.16).

In Section 4.3 branching ideals in rings are defined and studied in local real closed rings of finite rank, the main result here being Proposition 4.3.5, which gives equivalent characterizations for the maximal ideal in these rings to be a branching ideal. Section

4.3 can be read independently of Section 4.2.

In Section 4.4 the real closed version of Theorem 4.2.22 is proved (Theorem 4.4.2), and this is then used to formally define the main objects of this chapter, namely rings of type $(n, 1)$ and of type $(n, 2)$ (Lemma 4.4.7 and Definition 4.4.8); the section concludes by proving some embedding lemmas of such rings which are used in the model completeness proofs of Section 4.5.

All the model theory of the chapter is contained in Sections 4.5 and 4.6. Section 4.5 starts by defining all the relevant first-order theories, and this is followed by the model completeness results (Theorems 4.5.15 and 4.5.21) from which much of the remaining statements in Section 4.5 stem from. Section 4.6 starts by explaining some difficulties in the model-theoretic study of arbitrary local real closed SV-rings of finite rank. In Subsection 4.6.2 an approach to overcome these difficulties is proposed by introducing the notion of the branching spectrum of a local real closed ring of finite rank and connecting it with the model theory of real closed rings with a radical relation as developed in [PS]. In particular, Corollary 4.6.8 shows that elementary equivalent local real closed rings of finite rank have poset-isomorphic branching spectra, and this yields a candidate for an elementary classification of all local real closed SV-rings of finite rank (Conjecture 4.6.9).

4.2 SV-rings

4.2.1 Preliminaries on SV-rings

Definition 4.2.1. A ring A is an *SV-ring* if $A/\mathfrak{p}$ is a valuation ring for all $\mathfrak{p} \in \operatorname{Spec}(A)$.

In particular, by Theorem 2.3.2 (II) (i) a real closed SV-ring is a real closed ring A such that $A/\mathfrak{p}$ is a real closed valuation ring for all $\mathfrak{p} \in \operatorname{Spec}(A)$. Residue domains of valuation rings are valuation rings, therefore valuation rings are the first examples of SV-rings. Rings of Krull dimension 0 (i.e., rings in which every prime ideal is maximal) are clearly also SV-rings; for more examples of SV-rings see Proposition 4.2.4 and the subsections below. The next theorem collects some known equivalent characterizations of SV-rings:

Theorem 4.2.2. *Let A be a ring. The following are equivalent:*

(i) *A is an SV-ring.*

(ii) *$A/\mathfrak{p}$ is a valuation ring for all $\mathfrak{p} \in \mathrm{Spec}^{\min}(A)$.*

(iii) *$A_{\mathrm{red}} := A/\mathrm{Nil}(A)$ is an SV-ring, where $\mathrm{Nil}(A)$ is the nilradical of A.*

(iv) *$\mathrm{Spec}(A)$ is a normal space[1] and the localization $A_{\mathfrak{m}}$ is an SV-ring for every maximal ideal $\mathfrak{m}$ of A.*

(v) *For each $a, b \in A_{\mathrm{red}}$ there exists a polynomial $P \in A_{\mathrm{red}}[X,Y]$ of the form $P(X,Y) := \prod_{i=1}^{r}(X - c_i \cdot Y)$ such that $P(a,b) \cdot P(b,a) = 0$.*

Moreover, if A satisfies any of the conditions (i) - (vi) *above, then A_{red} is isomorphic to a subdirect product of valuation rings.*

Proof. The equivalence of (i) - (iii) is clear, the equivalence of (i) and (iv) is Proposition 1.5 in [Sch10b], and the equivalence of (i) and (v) follows from the equivalence of (i) and (iii) together with Theorem 3.4 in [Sch10b]. To conclude, suppose that A is an SV-ring and let $B := A_{\mathrm{red}}$; B is a reduced SV-ring by the implication (i) $\Rightarrow$ (iii), therefore the canonical map $B \longrightarrow \prod_{\mathfrak{p} \in \mathrm{Spec}(B)} B/\mathfrak{p}$ is injective and its image is a subdirect product of the valuation rings $\{B/\mathfrak{p}\}_{\mathfrak{p} \in \mathrm{Spec}(B)}$. □

It follows from the equivalences (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) in Theorem 4.2.2 that in the study of SV-rings, the class of reduced local SV-rings is at the forefront.

Remark 4.2.3. (i) Rings with normal Zariski spectrum are abundant in the realm of real algebra. In particular, the Zariski spectrum of a real closed ring is normal by Theorem 2.3.2 (II) (iii), and thus for such class of rings the SV property is a "local property" by the equivalence (i) $\Leftrightarrow$ (iv) in Theorem 4.2.2.

(ii) In general, subdirect products of valuation rings are not SV-rings: If X is not an SV-space (see [HW92]), then the ring $C(X)$ of continuous real-valued functions on X is a subdirect product of valuation rings which is not an SV-ring. On the other hand, every subdirect product[2] of finitely many valuation rings is an SV-ring (see Subsection 4.2.3 and Remark 4.2.23).

[1] Equivalently, A is a *Gelfand ring*; see [Joh82, p. 199] and [ST10, Theorem 4.3].

[2] The model theory of subdirect products of structures has been investigated via a 2-sorted set-up in [Wei75]; see also [Wei78].

One way to obtain examples of SV-rings to find ring-theoretic constructions which preserve the SV property; the next proposition summarizes the main such constructions:

Proposition 4.2.4. *The class of SV-rings is closed under formation of residue rings and localizations by multiplicative subsets. Moreover:*

(i) *A finite direct product of rings is an SV-ring if and only if each factor is an SV-ring.*

(ii) *Direct limits of SV-rings are SV-rings.*

(iii) *Direct products of valuation rings are SV-rings.*

(iv) *If A and B are SV-rings and $f : A \twoheadrightarrow C$ and $g : B \twoheadrightarrow C$ are surjective ring homomorphisms onto a ring C, then the fibre product $A \times_C B$ is an SV-ring.*

Proof. That the class of SV-rings is closed under the formation of residue rings and localizations by multiplicative subsets is clear, and item (i) follows from the characterization of prime ideals in a finite direct product of rings; for the proof of items (ii) - (iv) see Example 1.2 in [Sch10b]. □

In particular, if V_1 and V_2 are non-trivial valuation rings with isomorphic residue field $\boldsymbol{k}$, then by Proposition 4.2.4 (iv) the fibre product $V_1 \times_{\boldsymbol{k}} V_2$ is an SV-ring; $V_1 \times_{\boldsymbol{k}} V_2$ should be thought as constructed by "gluing" the valuation rings V_1 and V_2 in a particular way, and this construction of SV-rings via fibre products is a central theme of this chapter. The next example, which generalizes Proposition 4.2.4 (iii), shows how to construct some SV-rings by gluing valuations rings via a sheaf construction:

Example 4.2.5. Let A be a *Boolean product* of valuation rings in the language $\mathscr{L}(\text{div})$, where $\mathscr{L}(\text{div})$ is the language of rings $\mathscr{L} := \{+, -, \cdot, 0, 1\}$ together with a binary predicate div interpreted as divisibility; i.e., there exists a Boolean space X and a Hausdorff sheaf $\mathcal{O}_X$ of valuation rings on X (that is, the stalk $\mathcal{O}_{X,x}$ is a valuation ring for each $x \in X$) such that A is the $\mathscr{L}(\text{div})$-structure $\Gamma(X, \mathcal{O}_X)$ of global continuous sections, see [BW79] and [Gui01, Section 2]. It will be shown that A is an SV-ring; to this end, set $A_U := \Gamma(U, \mathcal{O}_X)$ for each $U \subseteq X$ open, $A_x := \mathcal{O}_{X,x}$ for each $x \in X$, and write $a_{\restriction U}$ for the image of $a \in A$ in A_U and $a(x)$ for the image of $a \in A$ in A_x. Pick

$a, b \in A$ and note that $X = U \dot\cup V \dot\cup W$, where $U := [\![\mathrm{div}(a,b)]\!] \cap (X \setminus [\![\mathrm{div}(b,a)]\!])$, $V := [\![\mathrm{div}(b,a)]\!] \cap (X \setminus [\![\mathrm{div}(a,b)]\!])$, $W := [\![\mathrm{div}(a,b)]\!] \cap [\![\mathrm{div}(b,a)]\!]$, and $[\![\mathrm{div}(a,b)]\!] := \{x \in X \mid A_x \models \mathrm{div}(a(x), b(x))\}$; since A is a Boolean product in the language $\mathscr{L}(\mathrm{div})$, $U, V, W \subseteq X$ are clopen, and a standard compactness argument (e.g. as in the proof of [BW79, Lemma 1.1]) yields elements $d \in A_U$, $e \in A_V$, and $f \in A_W$ such that $a_{\restriction U} d = b_{\restriction U}$, $b_{\restriction V} e = a_{\restriction V}$, and $b_{\restriction W} f = a_{\restriction W}$, hence $c := d \cup e \cup f \in A$ is an element such that $(a - cb)(b - ca) = 0$, therefore A is an SV-ring by the implication (v) $\Rightarrow$ (i) in Theorem 4.2.2.

4.2.2 The rank of a ring

One way of classifying the complexity of SV-rings is via the following notion of rank of a ring (this definition can be found in [Sch10b, Section 2]):

Definition 4.2.6. Let A be a ring and ∞ be a symbol such that $n < \infty$ for all $n \in \mathbb{N}$.

(i) For $\mathfrak{p} \in \mathrm{Spec}(A)$, define $\mathrm{rk}(A, \mathfrak{p}) \in \mathbb{N}$ to be the number of minimal prime ideals $\mathfrak{q}$ of A such that $\mathfrak{q} \subseteq \mathfrak{p}$ if this number is finite, and $\mathrm{rk}(A, \mathfrak{p}) = \infty$ otherwise.

(ii) The *rank of* A is $\mathrm{rk}(A) := \sup\{\mathrm{rk}(A, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\} \in \mathbb{N} \cup \{\infty\}$.

The ring A is *of finite rank* if $\mathrm{rk}(A) \neq \infty$.

It has been already noted that reduced local SV-rings are in the forefront of the study of SV-rings; amongst reduced local SV-rings, those of rank 1 are exactly the simplest SV-rings, namely valuation rings:

Lemma 4.2.7. *Let A be a ring. The following are equivalent:*

(i) *A is a reduced local SV-ring of rank 1.*

(ii) *A is a valuation ring.*

Proof. The implication (ii) $\Rightarrow$ (i) is trivial. Conversely, if item (i) holds, then A being local of rank 1 implies that A has exactly one minimal prime ideal $\mathfrak{q}$, therefore $\mathrm{Nil}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p} = \mathfrak{q}$ is a prime ideal of A; since A is reduced, $\mathfrak{q} = (0)$, and since A is an SV-ring, $A/\mathfrak{q} = A$ is a valuation ring, as required. $\square$

Corollary 4.2.8. *A ring A is a local real closed SV-ring of rank 1 if and only if it is a real closed valuation ring.*

Proof. Immediate by 4.2.7. □

The same proof as in Lemma 4.2.7 shows that local domains are exactly the reduced local rings of rank 1; in particular, if a reduced local ring is not a domain, then its rank is at least 2, so the rank of a reduced local ring is related with its zero divisors. The next lemmas clarify the relationship between the rank of a reduced local ring and its zero divisors (cf. [Sch10b, Proposition 2.1] and [Lar10, Theorem 3.6]); Lemma 4.2.9 (III) and Corollary 4.2.11 will be of particular importance in Section 4.5.

Lemma 4.2.9. *Let A be a reduced ring and set $\mathrm{Spec}^{\min}(A) := \{\mathfrak{p}_i \mid i \in I\}$.*

(I) *If $a \in A$ is a non-zero zero divisor, then $S := \{i \in I \mid a \notin \mathfrak{p}_i\}$ is a non-empty proper subset of I such that $\bigcap_{i\in S} \mathfrak{p}_i = \mathrm{Ann}(a)$.*

(II) *Let $m \in \mathbb{N}^{\geq 2}$. If $a_1, \dots, a_m \in A$ are non-zero and pairwise orthogonal[3], then $S_j := \{i \in I \mid a_j \notin \mathfrak{p}_i\}$ $(j \in [m])$ are pairwise disjoint non-empty proper subsets of I such that $\bigcap_{i\in S_j} \mathfrak{p}_i = \mathrm{Ann}(a_j)$ for all $j \in [m]$.*

(III) *Suppose that A is local.*

(i) $\mathrm{rk}(A) = \sup\{m \in \mathbb{N} \mid \exists a_1, \dots, a_m \in A$ non-zero and pairwise orthogonal$\}$.

(ii) *If $\mathrm{rk}(A) = |I| = n \in \mathbb{N}^{\geq 2}$, then for all $a_1, \dots, a_n \in A$ non-zero and pairwise orthogonal there exists a bijection $\sigma : [n] \longrightarrow [n]$ such that $\mathfrak{p}_i = \mathrm{Ann}(a_{\sigma(i)})$ for all $i \in [n]$.*

Proof. (I). Since A is reduced and $a \in A$ is a non-zero zero divisor, $a \notin \bigcap_{i\in I} \mathfrak{p}_i = (0)$ and $a \in \bigcup_{i\in I} \mathfrak{p}_i$, therefore $S := \{i \in I \mid a \notin \mathfrak{p}_i\}$ is a non-empty proper subset of I; note in particular that $a \in \mathfrak{p}_i$ for all $i \in I \setminus S$. The inclusion $\mathrm{Ann}(a) \subseteq \bigcap_{i\in S} \mathfrak{p}_i$ is clear; conversely, if $b \in \bigcap_{i\in S} \mathfrak{p}_i$, then $ba \in (\bigcap_{i\in S} \mathfrak{p}_i) \cap (\bigcap_{i\in I\setminus S} \mathfrak{p}_i) = (0)$, hence $b \in \mathrm{Ann}(a)$.

(II). Each a_j is a non-zero zero divisor, therefore by (I) it remains to show that the sets $S_1, \dots, S_m$ defined in (II) are pairwise disjoint. Assume for contradiction that there exist $j, j' \in [m]$ with $j \neq j'$ such that there exists $i \in S_j \cap S_{j'}$, i.e., $a_j, a_{j'} \notin \mathfrak{p}_i$; since $j \neq j'$, $a_j a_{j'} = 0 \in \mathfrak{p}_i$, giving the required contradiction.

[3] Two elements $a, b \in A$ in a ring A are *orthogonal* if $ab = 0$.

(III) (i). Note first that $\mathrm{rk}(A) = |\mathrm{Spec}^{\min}(A)| = |I| \in \mathbb{N} \cup \{\infty\}$ by assumption on A. Define $\kappa := \sup\{m \in \mathbb{N} \mid \exists a_1, \ldots, a_m \in A$ non-zero and pairwise orthogonal$\} \in \mathbb{N} \cup \{\infty\}$; if $\kappa \notin \mathbb{N}$, then $\mathrm{rk}(A) \notin \mathbb{N}$ by (II), therefore $\mathrm{rk}(A) = \kappa$, and if $\kappa = 1$, then A is a domain, therefore $\mathrm{rk}(A) = 1 = \kappa$. Suppose now that $\kappa = n \in \mathbb{N}^{\geq 2}$; let $a_1, \ldots, a_n \in A$ be non-zero pairwise orthogonal elements witnessing $\kappa = n$, and define the subsets $S_j \subseteq I$ as in item (II); by (II), $n \leq \mathrm{rk}(A)$, so assume for contradiction that $n < \mathrm{rk}(A)$.

Case 1. There exists $i \in I \setminus (S_1 \dot\cup \ldots \dot\cup S_n)$. In this case $\mathfrak{p}_i$ is a minimal prime ideal such that $a_1, \ldots, a_n \in \mathfrak{p}_i$; since $\mathfrak{p}_i$ is a minimal prime ideal, by [Mat83, Proposition 1.2 (1)] there exists $b_j \in A \setminus \mathfrak{p}_i$ such that $a_j b_j = 0$ for each $j \in [n]$, therefore $b := b_1 \cdot \ldots \cdot b_n$ is a non-zero element which is orthogonal to each a_i, a contradiction to $\kappa = n$.

Case 2. $I = S_1 \dot\cup \ldots \dot\cup S_n$ and there exists $j \in [n]$ such that $|S_j| \geq 2$. Assume without loss of generality that $|S_1| \geq 2$ and suppose that $\mathfrak{p}_{i_1}$ and $\mathfrak{p}_{i_2}$ are distinct minimal prime ideals such that $i_1, i_2 \in S_1$, so that $a_1 \notin \mathfrak{p}_{i_1}$ and $a_1 \notin \mathfrak{p}_{i_2}$. Pick $b \in \mathfrak{p}_{i_1} \setminus \mathfrak{p}_{i_2}$; since $\mathfrak{p}_{i_1}$ is a minimal prime ideal, by [Mat83, Proposition 1.2 (1)] there exists $c \in A \setminus \mathfrak{p}_{i_1}$ such that $bc = 0$, therefore $a_1 b, a_1 c, a_2, \ldots, a_n$ are non-zero pairwise orthogonal elements of A, a contradiction to $\kappa = n$.

(III) (ii). Combine items (II) and (III) (i). □

Remark 4.2.10. Let A be a reduced local ring of rank $n \in \mathbb{N}^{\geq 2}$ and write $\mathrm{Spec}^{\min}(A) := \{\mathfrak{p}_i \mid i \in [n]\}$; the canonical map $A \longrightarrow \prod_{i \in [n]} A/\mathfrak{p}_i$ is an embedding, and it follows from this that (up to re-labelling) any n non-zero pairwise orthogonal elements $a_1 \ldots, a_n \in A$ satisfy $a_i \in \bigcap_{j \in [n] \setminus \{i\}} \mathfrak{p}_j \setminus \mathfrak{p}_i$, so that $\mathfrak{p}_i = \mathrm{Ann}(a_i)$ for all $i \in [n]$.

Corollary 4.2.11. *Let A and B be reduced local rings of rank $n \in \mathbb{N}^{\geq 2}$ such that $A \subseteq B$. If $a_1, \ldots, a_n \in A$ are non-zero and pairwise orthogonal, then*

$$\mathrm{Spec}^{\min}(A) = \{\mathrm{Ann}_A(a_i) \mid i \in [n]\} \quad \textit{and} \quad \mathrm{Spec}^{\min}(B) = \{\mathrm{Ann}_B(a_i) \mid i \in [n]\}.$$

Proof. Clear by Lemma 4.2.9 (III) (ii). □

Remark 4.2.12. Let A and B be reduced local rings of rank $n \in \mathbb{N}^{\geq 2}$ such that $A \subseteq B$. Write $\mathrm{Spec}^{\min}(A) = \{\mathfrak{p}_{A,i} \mid i \in [n]\}$ and $\mathrm{Spec}^{\min}(B) = \{\mathfrak{p}_{B,i} \mid i \in [n]\}$, and assume without loss of generality that $\mathfrak{p}_{B,i} \cap A = \mathfrak{p}_{A,i}$ for all $i \in [n]$ (Corollary 4.2.11). The embedding $A \subseteq B$ induces embeddings $A/\mathfrak{p}_{A,i} \subseteq B/\mathfrak{p}_{B,i}$ for all $i \in [n]$, and $A \subseteq B$ is a local embedding if and only if $A/\mathfrak{p}_{A,i} \subseteq B/\mathfrak{p}_{B,i}$ is a local embedding for all (equivalently, for some) $i \in [n]$: this follows from the fact that the residue field of A is isomorphic to the residue field of $A/\mathfrak{p}_{A,i}$ for all $i \in [n]$, and similarly for B.

Lemma 4.2.13. *Let $n \in \mathbb{N}^{\geq 2}$, $A_1, \ldots, A_n$ be domains, and A be a subdirect product of $\{A_1, \ldots, A_n\}$.*

(I) *For every $\mathfrak{p} \in \operatorname{Spec}^{\min}(A)$ there exists $i \in [n]$ such that $\mathfrak{p} = \ker(p_i)$ (see the beginning of this chapter for the definition of p_i); in particular, $\operatorname{rk}(A) \leq n$.*

(II) *Suppose that A is local, and for each $S \subseteq [n]$ let $\pi_S : \prod_{i=1}^{n} A_i \twoheadrightarrow \prod_{i \in S} A_i$ be the canonical projection. The following are equivalent:*

  (i) $\operatorname{rk}(A) = n$.

  (ii) *For all $S \subsetneq [n]$, the map $p_S := (\pi_S)_{\restriction A} : A \longrightarrow \prod_{i \in S} A_i$ is not injective.*

  (iii) $\ker(p_i)$ *and* $\ker(p_j)$ *are incomparable under subset inclusion for all $i, j \in [n]$ with $i \neq j$.*

Moreover, if any of the conditions (i) - (iii) *holds, then* $\operatorname{Spec}^{\min}(A) = \{\ker(p_i) \mid i \in [n]\}$.

Proof. (I). Let $\mathfrak{p} \in \operatorname{Spec}^{\min}(A)$. Then $\ker(p_1) \cdot \ldots \cdot \ker(p_n) = (0) \subseteq \mathfrak{p}$, therefore there exists $i \in [n]$ such that $\ker(p_i) \subseteq \mathfrak{p}$; but $\ker(p_i) \in \operatorname{Spec}(A)$, hence $\ker(p_i) = \mathfrak{p}$ by minimality of $\mathfrak{p}$.

(II). Straightforward using (I). □

4.2.3 Structure theorem for reduced local SV-rings of finite rank

The main statement in this subsection is Theorem 4.2.22, which is just a refined formulation of the observation made in [Sch10b, Remark 3.2] that every reduced local SV-ring of finite rank can be constructed from valuation rings using iterated fibre products.

Lemma 4.2.14. *Let A be a ring and $I, J \subseteq A$ be ideals. The ring homomorphism $f : A \longrightarrow A/I \times A/J$ given by $f(a) := (a/I, a/J)$ has image $\frac{A}{I} \times_{\frac{A}{I+J}} \frac{A}{J}$; in particular, $\frac{A}{I \cap J} \cong \frac{A}{I} \times_{\frac{A}{I+J}} \frac{A}{J}$.*

Proof. Clearly $\operatorname{im}(f) \subseteq \frac{A}{I} \times_{\frac{A}{I+J}} \frac{A}{J}$; moreover, if $(a/I, b/J) \in \frac{A}{I} \times_{\frac{A}{I+J}} \frac{A}{J}$, then there exist $c \in I$ and $d \in J$ such that $a - b = c + d$, hence $a - c = b + d$ and thus $(a/I, b/J) = f(a - c) = f(b - d)$, therefore $\operatorname{im}(f) = \frac{A}{I} \times_{\frac{A}{I+J}} \frac{A}{J}$. The last assertion follows by noting that $\ker(f) = I \cap J$. □

Corollary 4.2.15. *Let $n \in \mathbb{N}^{\geq 2}$. If $I_1, \ldots, I_n \subseteq A$ are ideals, then*

$$\frac{A}{\bigcap_{i=1}^{n} I_i} \cong \left(\left(\left(\frac{A}{I_1} \times_{\frac{A}{G_2}} \frac{A}{I_2} \right) \times_{\frac{A}{G_3}} \frac{A}{I_3} \right) \cdots \times_{\frac{A}{G_{n-1}}} \frac{A}{I_{n-1}} \right) \times_{\frac{A}{G_n}} \frac{A}{I_n},$$

where $G_j := (\bigcap_{i=1}^{j-1} I_i) + I_j$ for all $j \in \{2, \ldots, n\}$.

Proof. Straightforward by induction using Lemma 4.2.14. □

Lemma 4.2.16 (Goursat's lemma for rings)**.** *Let A_1 and A_2 be rings and $A \subseteq A_1 \times A_2$ be a subring. The following are equivalent:*

(i) *A is a subdirect product of $\{A_1, A_2\}$.*

(ii) *There exist ideals $I \subseteq A_1$, $J \subseteq A_2$, and an isomorphism $f : A_1/I \xrightarrow{\cong} A_2/J$ such that $A = A_1 \times_{A_2/J} A_2$.*

(iii) *There exist ideals $H_1, H_2 \subseteq A$ such that $H_1 \cap H_2 = (0)$, together with isomorphisms $g_i : A_i \xrightarrow{\cong} A/H_i$ ($i \in \{1, 2\}$) yielding an isomorphism $g : A_1 \times A_2 \xrightarrow{\cong} A/H_1 \times A/H_2$ such that $g_{\restriction A}(a) = (a/H_1, a/H_2)$ for all $a \in A$, and $g_{\restriction A}$ is an isomorphism $A \xrightarrow{\cong} \frac{A}{H_1} \times_{\frac{A}{H_1 \oplus H_2}} \frac{A}{H_2}$.*

Proof. (i) ⇒ (ii). The following proof can be found in [Lar11, Lemma 2]; for future reference it is included here. Let $I := p_1(\ker(p_2))$ and $J := p_2(\ker(p_1))$, noting that since p_1 and p_2 are surjective, $I \subseteq A_2$ and $J \subseteq A_2$ are ideals (see the beginning of this chapter for the definition of p_i).

Claim. The assignment $f : A_1/I \longrightarrow A_2/J$ given by

$$a_1/I \mapsto a_2/J \overset{\text{def}}{\iff} (a_1, a_2) \in A$$

for all $(a_1, a_2) \in A_1 \times A_2$ is a well-defined ring isomorphism.

Proof of Claim. To see that f is indeed a function, pick $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$ such that $a_1 - b_1 \in I$ and $(a_1, a_2), (b_1, b_2) \in A$; then $(a_1 - b_1, a_2 - b_2) \in A$, and since $a_1 - b_1 \in I = p_1(\ker(p_2))$, $(a_1 - b_1, 0) \in A$, hence $0/J = f((a_1 - b_1)/I) = (a_2 - b_2)/J$, from which $a_2/J = b_2/J$ follows. That f is a ring homomorphism follows from A being a subring of $A_1 \times A_2$, and since A is a subdirect product of $\{A_1, A_2\}$ it follows that f is surjective, so it remains to show that f is injective. Pick $a_1, b_1 \in A_1$ and suppose that $f(a_1/I) = f(b_1/I)$, hence there exist $a_2, b_2 \in A_2$ with $a_2 - b_2 \in J$ and $(a_1, a_2), (b_1, b_2) \in A$; then $(a_1 - b_1, 0) = (a_1, a_2) - (b_1, b_2) - (0, a_2 - b_2) \in A$, yielding $a_1/I = b_1/I$. $\square_{\text{Claim}}$

Let $q_I : A_1 \twoheadrightarrow A_1/I$ and $q_J : A_2 \twoheadrightarrow A_2/J$ be the projection maps. By the claim above, $(f \circ q_I) \circ p_1 = q_J \circ p_2$, so to show that $A = A_1 \times_{A_2/J} A_2$, it suffices to show that A verifies the universal property of the fibre product $A_1 \times_{A_2/J} A_2$. Let $f_i : C \longrightarrow A_i$ be ring homomorphisms such that $(f \circ q_I) \circ f_1 = q_J \circ f_2$, and define $h : C \longrightarrow A_1 \times A_2$ by $h(c) := (f_1(c), f_2(c))$; since $f(f_1(c)/I) = f_2(c)/J$ by choice of f_1 and f_2, it follows by definition of f that $(f_1(c), f_2(c)) \in A$ for all $c \in C$, so $h(C) \subseteq A$ and thus the corestriction of h to A verifies the universal property of $A_1 \times_{A_2/J} A_2$.

<u>(ii) $\Rightarrow$ (iii).</u> Let $q_I : A_1 \twoheadrightarrow A_1/I$ and $q_J : A_2 \longrightarrow A_2/J$ be the projection maps. Since $f \circ q_I : A_1 \twoheadrightarrow A_2/J$ and $q_J : A_2 \twoheadrightarrow A_2/J$ are surjective and $A = A_1 \times_{A_2/J} A_2$, $p_i := \pi_{i\restriction A} : A \twoheadrightarrow A_i$ $(i \in \{1, 2\})$ are surjective ring homomorphisms such that $(f \circ q_I) \circ p_1 = q_J \circ p_2$. Set $H_i := \ker(p_i)$ for $i \in \{1, 2\}$; then $H_1, H_2 \subseteq A$ are ideals of A such that $H_1 \cap H_2 = (0)$ which induce isomorphisms $g_i : A_i \xrightarrow{\cong} A/H_i$ $(i \in \{1, 2\})$ yielding an isomorphism $g : A_1 \times A_2 \xrightarrow{\cong} A/H_1 \times A/H_2$ given by $(a_1, a_2) \mapsto (g_1(a_1), g_2(a_2))$. If $a := (a_1, a_2) \in A \subseteq A_1 \times A_2$, then $g(a) = (a/H_1, a/H_2)$ by definition of g, therefore $g_{\restriction A} : A \longrightarrow A/H_1 \times A/H_2$ is an isomorphism onto its image $\frac{A}{H_1} \times_{\frac{A}{H_1 \oplus H_2}} \frac{A}{H_2}$ by Lemma 4.2.14.

<u>(iii) $\Rightarrow$ (i).</u> Let $i \in \{1, 2\}$. By (iii), the following diagram

$$\begin{array}{ccccc}
A & \overset{\subseteq}{\hookrightarrow} & A_1 \times A_2 & \twoheadrightarrow & A_i \\
\downarrow{\scriptstyle g_{\restriction A}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_i} \\
\frac{A}{H_1} \times_{\frac{A}{H_1 \oplus H_2}} \frac{A}{H_2} & \overset{\subseteq}{\hookrightarrow} & \frac{A}{H_1} \times \frac{A}{H_2} & \twoheadrightarrow & A/H_i
\end{array}$$

commutes; since all the vertical arrows are isomorphisms, the composite top morphism is surjective if and only if the composite bottom morphism is surjective; since $\frac{A}{H_1} \times_{\frac{A}{H_1 \oplus H_2}} \frac{A}{H_2}$ is a subdirect product of $\{A/H_1, A/H_2\}$, (i) follows. $\square$

Remark 4.2.17. Let A_1 and A_2 be rings and $A \subseteq A_1 \times A_2$ be a subring satisfying any of the items (i) - (iii) in Lemma 4.2.16; then $\frac{A}{H_1 \oplus H_2} \cong A_1/I$ ($\cong A_2/J$), where $I \subseteq A_1$ and $J \subseteq A_2$ are ideals as in item (ii) in Lemma 4.2.16, and $H_1, H_2 \subseteq A$ are ideals as in item (iii) in Lemma 4.2.16. Indeed, by the proofs of the implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) in Lemma 4.2.16, $I = p_1(\ker(p_2))$ and $H_i = \ker(p_i)$ for $i \in \{1, 2\}$, hence $I = p_1(H_2)$; it is easy to check from this that $\frac{a}{H_1 \oplus H_2} \mapsto p_1(a)/I$ is a well-defined isomorphism $\frac{A}{H_1 \oplus H_2} \longrightarrow A_1/I$.

Corollary 4.2.18. *Let A_1 and A_2 be local rings and $A \subseteq A_1 \times A_2$ be a subring. The following are equivalent:*

(i) *A is a local ring and a subdirect product of $\{A_1, A_2\}$.*

(ii) *There exist ideals $I \subsetneq A_1$ and $J \subsetneq A_2$ and a ring isomorphism $f : A_1/I \xrightarrow{\cong} A_2/J$ such that $A = A_1 \times_{A_2/J} A_2$.*

(iii) *There exist ideals $H_1, H_2 \subseteq A$ such that $H_1 \cap H_2 = (0)$ and $H_1 \oplus H_2 \neq A$, together with isomorphisms $g_i : A_i \xrightarrow{\cong} A/H_i$ ($i \in \{1, 2\}$) yielding an isomorphism $g : A_1 \times A_2 \xrightarrow{\cong} A/H_1 \times A/H_2$ such that $g_{\restriction A}(a) = (a/H_1, a/H_2)$ for all $a \in A$, and $g_{\restriction A}$ is an isomorphism $A \xrightarrow{\cong} \frac{A}{H_1} \times_{\frac{A}{H_1 \oplus H_2}} \frac{A}{H_2}$.*

Moreover, if any of the items (i) - (iii) *holds, then one may choose:*

(a) *$I := p_1(\ker(p_2))$ and $J := p_2(\ker(p_1))$ to be the ideals of A_1 and A_2 in item* (ii) *(respectively) and $f : A_1/I \longrightarrow A_2/J$ to be the map $a_1/I \mapsto a_2/J$; and*

(b) *$H_i := \ker(p_i)$ to be the ideals of A ($i \in \{1, 2\}$) in item* (iii)*, and $g_i : A_i \longrightarrow A/H_i$ to be the corresponding induced isomorphisms.*

Proof. The equivalence of items (i) - (iii) is immediate from Lemma 4.2.16 noting that if B and C are local rings and $g : B \twoheadrightarrow D$ and $h : C \twoheadrightarrow D$ are surjective ring homomorphisms, then $B \times_D C$ is a local ring if and only if D is not the zero ring. The last statement in the lemma follows from the proofs of the implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) in Lemma 4.2.16. ∎

Notation 4.2.19. Let $n \in \mathbb{N}^{\geq 2}$ and suppose that $A_1, \ldots, A_n$ are rings. For each $k \in [n]$ and $j \in [k]$, let $\pi^k_j : \prod_{i=1}^k A_i \twoheadrightarrow \prod_{i=1}^j A_i$ be the projection map onto the first j factors; in particular, $\pi^n_1 = \pi_1$ and π^n_n is the identity map.

Proposition 4.2.20. *Let $n \in \mathbb{N}^{\geq 2}$, $A_1, \ldots, A_n$ be local rings, and $A \subseteq \prod_{i=1}^n A_i$ be a subring. The following are equivalent:*

(i) *A is a local ring and a subdirect product of $\{A_1, \ldots, A_n\}$.*

(ii) *$\pi_1^n(A) = A_1$ (see* Notation 4.2.19*) and for all $j \in [n-1]$ there exist ideals $I_j \subsetneq \pi_j^n(A)$ and $J_{j+1} \subsetneq A_{j+1}$ such that $\pi_j^n(A)/I_j \cong A_{j+1}/J_{j+1}$ and*

$$\pi_{j+1}^n(A) = \pi_j^n(A) \times_{A_{j+1}/J_{j+1}} A_{j+1};$$

in particular,

$$\pi_n^n(A) = A = (((A_1 \times_{A_2/J_2} A_2) \times_{A_3/J_3} A_3) \cdots \times_{A_{n-1}/J_{n-1}} A_{n-1}) \times_{A_n/J_n} A_n.$$

(iii) *There exist ideals $H_1, \ldots, H_n \subseteq A$ with $\bigcap_{i=1}^n H_i = (0)$ and $G_j := (\bigcap_{i=1}^{j-1} H_i) + H_j \neq A$ for all $j \in \{2, \ldots, n\}$, together with isomorphisms $g_i : A_i \xrightarrow{\cong} A/H_i$ ($i \in [n]$) yielding an isomorphism $g : \prod_{i=1}^n A_i \xrightarrow{\cong} \prod_{i=1}^n A/H_i$ such that $g_{\restriction A}(a) = (a/H_1, \ldots, a/H_n)$, and $g_{\restriction A}$ is an isomorphism*

$$A \cong \left(\left(\left(\frac{A}{H_1} \times_{\frac{A}{G_2}} \frac{A}{H_2}\right) \times_{\frac{A}{G_3}} \frac{A}{H_3}\right) \cdots \times_{\frac{A}{G_{n-1}}} \frac{A}{H_{n-1}}\right) \times_{\frac{A}{G_n}} \frac{A}{H_n}.$$

Proof. The proof is by induction on $n \in \mathbb{N}^{\geq 2}$. The base case $n = 2$ is exactly Corollary 4.2.18, so assume now that the proposition holds true for some $n := k \in \mathbb{N}^{>2}$. Let $A_1, \ldots, A_k, A_{k+1}$ be local rings, $A \subseteq \prod_{i=1}^{k+1} A_i$ be a subring, and define $B := \pi_k^{k+1}(A) \subseteq \prod_{i=1}^k A_i$; note in particular that $\pi_j^k(B) = \pi_j^{k+1}(A)$ for all $j \in [k]$.

<u>(i)⇒ (ii).</u> Since A is a subdirect product of $\{A_1, \ldots, A_k, A_{k+1}\}$, $\pi_1^{k+1}(A) = A_1$ and B is a subdirect product of $\{A_1, \ldots, A_k\}$, and since B is a homomorphic image of A and A is local, B is also local, hence by inductive hypothesis, for all $j \in [k-1]$ there exist ideals $I_j \subsetneq \pi_j^k(B)$ and $J_{j+1} \subsetneq A_{j+1}$ such that $\pi_j^k(B)/I_j \cong A_{j+1}/J_{j+1}$ and $\pi_{j+1}^k(B) = \pi_j^k(B) \times_{A_{j+1}/J_{j+1}} A_{j+1}$; since A is also a subdirect product of $\{B, A_{k+1}\}$ and B is a local, by the implication (i) ⇒ (ii) in Corollary 4.2.18 there exist ideals $I_k \subsetneq B$ and $J_{k+1} \subsetneq A_{k+1}$ such that $B/I_k \cong A_{k+1}/I_{k+1}$ and $A = B \times_{A_{k+1}/J_{k+1}} A_{k+1}$; (ii) now follows from $\pi_j^k(B) = \pi_j^{k+1}(A)$ for all $j \in [k-1]$ and $B = \pi_k^{k+1}(A)$.

<u>(ii) ⇒ (iii).</u> Since $\pi_1^{k+1}(A) = A_1$ by assumption, $\pi_1^k(B) = A_1$. Therefore, by inductive hypothesis there exist ideals $H_1', \ldots, H_k' \subseteq B$ such that $\bigcap_{i=1}^k H_i' = (0)$, $G_j' := (\bigcap_{i=1}^{j-1} H_i') + H_j' \neq B$ for all $j \in \{2, \ldots, k\}$, and there exist isomorphisms

$g'_i : A_i \xrightarrow{\cong} B/H'_i$ ($i \in [k]$) yielding an isomorphism $g' : \prod_{i=1}^k A_i \longrightarrow \prod_{i=1}^k B/H'_i$ such that $(g')_{\restriction B}(b) = (b/H'_1, \ldots, b/H'_k)$ for all $b \in B$, and $(g')_{\restriction B}$ is an isomorphism

$$B \cong \left(\left(\left(\frac{B}{H'_1} \times_{\frac{B}{G'_2}} \frac{B}{H'_2}\right) \times_{\frac{B}{G'_3}} \frac{B}{H'_3}\right) \cdots \times_{\frac{B}{G'_{k-1}}} \frac{B}{H'_{k-1}}\right) \times_{\frac{B}{G'_k}} \frac{B}{H'_k};$$

note in particular that since $A_i \cong B/H'_i$ is a local ring for all $i \in [k]$ and $G'_j \neq B$ for all $j \in \{2, \ldots, k\}$, B is also a local ring. Since $A = \pi_{k+1}^{k+1}(A) = \pi_k^{k+1}(A) \times_{A_{k+1}/J_{k+1}} A_{k+1} = B \times_{A_{k+1}/J_{k+1}} A_{k+1}$ by assumption, by the implication (ii) $\Rightarrow$ (iii) in Corollary 4.2.18 there exist ideals $H''_1, H''_2 \subseteq A$ such that $H''_1 \cap H''_2 = (0)$ and $H''_1 \oplus H''_1 \neq A$, and there exist isomorphisms $g''_1 : B \xrightarrow{\cong} A/H''_1$ and $g''_2 : A_{k+1} \xrightarrow{\cong} A/H''_2$ yielding an isomorphism $g'' : B \times A_{k+1} \longrightarrow A/H''_1 \times A/H''_2$ such that $(g'')_{\restriction A}(a) = (a/H''_1, a/H''_2)$ for all $a \in A$, and $(g'')_{\restriction A}$ is an isomorphism $A \cong A/H''_1 \times_{A/(H''_1 \oplus H''_2)} A/H''_2$; moreover, by Corollary 4.2.18 (b), $H''_1 = \ker((\pi_k^{k+1})_{\restriction A})$ and $H''_2 = \ker((\pi_{k+1})_{\restriction A}))$. Define $H_i := ((\pi_k^{k+1})_{\restriction A})^{-1}(H'_i)$ for all $i \in [k]$, $H_{k+1} := H''_2$, and $G_j := (\bigcap_{i=1}^{j-1} H_i) + H_j$ for all $j \in \{2, \ldots, k+1\}$. Then

$$\begin{aligned}\bigcap_{i=1}^k H_i &= \bigcap_{i=1}^k ((\pi_k^{k+1})_{\restriction A})^{-1}(H'_i) = ((\pi_k^{k+1})_{\restriction A})^{-1}\left(\bigcap_{i=1}^k H'_i\right) \\ &= ((\pi_k^{k+1})_{\restriction A})^{-1}(0) = \ker((\pi_k^{k+1})_{\restriction A}) = H''_1\end{aligned}$$

hence $\bigcap_{i=1}^{k+1} H_i = H''_1 \cap H''_2 = (0)$ and $G_{k+1} = (\bigcap_{i=1}^k H_i) + H_{k+1} = H''_1 \oplus H''_2 \neq A$. Moreover, since $(\pi_k^{k+1})_{\restriction A} : A \twoheadrightarrow B$ is surjective, it follows from the above that $(\pi_k^{k+1})_{\restriction A}$ induces isomorphisms $h_i : B/H'_i \longrightarrow A/H_i$ given by $\pi_k^{k+1}(a)/H'_i \mapsto a/H_i$ for all $i \in [k]$, and also that $G_j = ((\pi_k^{k+1})_{\restriction A})^{-1}(G'_j)$ and $A/G_j \cong B/G'_j$ for all $j \in \{2, \ldots, k\}$, hence $G_j \neq A$ for all $j \in \{2, \ldots, k\}$, and the isomorphisms $g_i := (h_i \circ g'_i) : A_i \longrightarrow A/H_i$ ($i \in [k]$) and $g_{k+1} := g''_2 : A_{k+1} \longrightarrow A/H_{k+1}$ yield an isomorphism $g : \prod_{i=1}^{k+1} A_i \longrightarrow \prod_{i=1}^{k+1} A/H_i$ verifying the last statement in item (iii).

(iii) $\Rightarrow$ (i). Clear using the same argument as in the proof of the implication (iii) $\Rightarrow$ (i) in Lemma 4.2.16 and noting that A is a local ring since A/H_i is a local ring for all $i \in [k+1]$ and $G_j \neq A$ for all $j \in \{2 \ldots, k+1\}$. □

Remark 4.2.21. The condition $\pi_1^n(A) = A_1$ is necessary for the equivalence of (i) and (ii) in Proposition 4.2.20 to be true. For example, let V be a non-trivial valuation ring, and define $A_1 := \mathrm{qf}(V)$, $A_2 := V$, and $A := V \times_{V/\mathfrak{m}_V} V$; then $\pi_1^2(A) = V \neq A_1$ and A is a subring of $A_1 \times A_2$ such that $A = \pi_1^2(A) \times_{A_2/J_2} A_2$ where $I_1 = J_2 := \mathfrak{m}_V$, but A it is not a subdirect product of $\{A_1, A_2\}$.

Theorem 4.2.22. *Let $n \in \mathbb{N}^{\geq 2}$ and A be a ring which is not a field. The following are equivalent:*

(i) *A is a reduced local SV-ring of rank at most n.*

(ii) *A is a local ring, and there exist non-trivial valuation rings $A_1, \dots, A_n$ and an injective ring homomorphism $\varepsilon : A \hookrightarrow \prod_{i=1}^n A_i$ such that $\varepsilon(A)$ is a subdirect product of $\{A_1, \dots, A_n\}$.*

(iii) *There exist non-trivial valuation rings $A_1, \dots, A_n$ and an injective ring homomorphism $\varepsilon : A \hookrightarrow \prod_{i=1}^n A_i$, such that $(\pi_1^n \circ \varepsilon)(A) = A_1$ (see* Notation 4.2.19*), and for all $j \in [n-1]$ there exist ideals $I_j \subsetneq (\pi_j^n \circ \varepsilon)(A)$ and $J_{j+1} \subsetneq A_{j+1}$ such that $(\pi_j^n \circ \varepsilon)(A)/I_j \cong A_{j+1}/J_{j+1}$ and*

$$(\pi_{j+1}^n \circ \varepsilon)(A) = (\pi_j^n \circ \varepsilon)(A) \times_{A_{j+1}/J_{j+1}} A_{j+1};$$

in particular, A is isomorphic to

$$(\pi_n^n \circ \varepsilon)(A) = (((A_1 \times_{A_2/J_2} A_2) \times_{A_3/J_3} A_3) \cdots \times_{A_{n-1}/J_{n-1}} A_{n-1}) \times_{A_n/J_n} A_n.$$

(iv) *There are prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subseteq A$ such that $\bigcap_{i=1}^n \mathfrak{p}_i = (0)$, $G_j := (\bigcap_{i=1}^{j-1} \mathfrak{p}_i) + \mathfrak{p}_j \neq A$ for all $j \in \{2, \dots, n\}$, $A/\mathfrak{p}_i$ is a non-trivial valuation ring for all $i \in [n]$, and the canonical ring homomorphism $A \longrightarrow \prod_{i=1}^n A/\mathfrak{p}_i$ given by $a \mapsto (a/\mathfrak{p}_1, \dots, a/\mathfrak{p}_n)$ restricts to an isomorphism*

$$A \cong \left(\left(\left(\frac{A}{\mathfrak{p}_1} \times_{\frac{A}{G_2}} \frac{A}{\mathfrak{p}_2} \right) \times_{\frac{A}{G_3}} \frac{A}{\mathfrak{p}_3} \right) \cdots \times_{\frac{A}{G_{n-1}}} \frac{A}{\mathfrak{p}_{n-1}} \right) \times_{\frac{A}{G_n}} \frac{A}{\mathfrak{p}_n}.$$

Moreover, if any of the items (i) - (iv) *holds, then the following are equivalent:*

(a) *A has rank exactly n.*

(b) *For all $S \subsetneq [n]$, the map $\pi_S \circ \varepsilon : A \twoheadrightarrow \prod_{i \in S} A_i$ is not injective, where A_i ($i \in [n]$) and ε are as in items* (ii) *and* (iii)*, and $\pi_S : \prod_{i=1}^n A_i \twoheadrightarrow \prod_{i \in S} A_i$ is the canonical projection.*

(c) *$\mathfrak{p}_i$ and $\mathfrak{p}_j$ are incomparable under subset inclusion for all $i, j \in [n]$ with $i \neq j$, where $\mathfrak{p}_i$ are the prime ideals of A in item* (iv)*.*

Proof. The equivalence of items (ii) - (iv) follows from Proposition 4.2.20; it therefore remains to show the equivalence of items (i) and (ii), as well as the equivalence of (a) - (c).

(i) $\Rightarrow$ (ii). Since A is local with rank at most n, $\mathrm{Spec}^{\min}(A) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ for some $m \in [n]$, and since A is reduced, $\mathrm{Nil}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}^{\min}(A)} \mathfrak{p} = (0)$, and thus the canonical map $A \longrightarrow \prod_{i=1}^{m} A/\mathfrak{p}_i$ is injective. Since A is not a field and a local SV-ring, each factor ring $A_i := A/\mathfrak{p}_i$ is a non-trivial valuation domain, therefore (ii) follows by taking $A_{m+1} = \cdots = A_n := A_m$ and $\varepsilon : A \hookrightarrow \prod_{i=1}^{n} A_i$ to be the canonical map $a \mapsto (a/\mathfrak{p}_1, \ldots, a/\mathfrak{p}_m, a/\mathfrak{p}_m, \ldots, a/\mathfrak{p}_m)$.

(ii) $\Rightarrow$ (i). Since each A_i is a domain and A is isomorphic to the subring $\varepsilon(A) \subseteq \prod_{i=1}^{n} A_i$, A is reduced, so it remains to show that A is an SV-ring of rank at most n. Since A is local, it suffices to show by the implication (ii) $\Rightarrow$ (i) in Theorem 4.2.2 that A has at most n minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, and that $A/\mathfrak{p}_i$ is a non-trivial valuation ring for all $i \in [n]$. For each $i \in [n]$, define $\mathfrak{p}_i := \ker(\pi_i \circ \varepsilon)$; then $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are prime ideals of A such that each $A/\mathfrak{p}_i$ is a non-trivial valuation ring, and it follows by Lemma 4.2.13 (I) that $\mathrm{Spec}^{\min}(A) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, as required.

If any of the items (i) - (iv) holds, then the equivalence of items (a) - (c) follows by Lemma 4.2.13 (II). □

Remark 4.2.23. Recall that a ring is *semi-local* if it has finitely many maximal ideals; it is not difficult to see from the proofs of Proposition 4.2.20 and of Theorem 4.2.22 that the class of rings which are isomorphic to subdirect products of finitely many local domains (resp., valuation rings) is exactly the class of reduced semi-local rings (resp., reduced semi-local SV-rings) of finite rank.

4.3 Branching ideals

Definition 4.3.1. Let A be a ring. A prime ideal $\mathfrak{q}$ of A (prime ideals are always proper subsets) is a *branching ideal* if there exist distinct $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(A)$ such that $\mathfrak{q}_1, \mathfrak{q}_2 \subsetneq \mathfrak{q}$ and $\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2$.

Remark 4.3.2. Let A be a real closed ring.

(i) If $\mathfrak{q} \in \mathrm{Spec}(A)$ is a branching ideal, then $\mathrm{rk}(A, \mathfrak{q}) \geq 2$ by Theorem 2.3.2 (II) (iii), but the converse does not hold: consider the maximal ideal of $A := V \times_{V/\mathfrak{p}} V$, where V is a non-trivial real closed valuation ring of Krull dimension at least 2 and $\mathfrak{p}$ is a non-zero non-maximal prime ideal of V, noting that A is a real closed ring by items (I) and (II) (i) in Theorem 2.3.2.

(ii) A has at least one branching ideal if and only if $\mathrm{rk}(A) \geq 2$. One implication is clear; conversely, if $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(A)$ witness that $\mathfrak{q} \in \mathrm{Spec}(A)$ is a branching ideal, then any $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}^{\min}(A)$ such that $\mathfrak{p}_1 \subseteq \mathfrak{q}_1$ and $\mathfrak{p}_2 \subseteq \mathfrak{q}_2$ are distinct by Theorem 2.3.2 (II) (iii), therefore $\mathrm{rk}(A) \geq \mathrm{rk}(A, \mathfrak{q}) \geq 2$.

Lemma 4.3.3. *Let A be a ring and let $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(A)$ be incomparable prime ideals in $(\mathrm{Spec}(A), \subseteq)$. If $\mathfrak{p}_1 \in \mathfrak{q}_1^{\downarrow}$ and $\mathfrak{p}_2 \in \mathfrak{q}_2^{\downarrow}$ are such that $\mathfrak{p}_1^{\uparrow}$ and $\mathfrak{p}_2^{\uparrow}$ are chains in $(\mathrm{Spec}(A), \subseteq)$ and $\mathfrak{p}_1 + \mathfrak{p}_2 \in \mathrm{Spec}(A)$, then $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{q}_1 + \mathfrak{q}_2$.*

Proof. The inclusion $\mathfrak{p}_1 + \mathfrak{p}_2 \subseteq \mathfrak{q}_1 + \mathfrak{q}_2$ is clear, and for the other inclusion it suffices to show that $\mathfrak{q}_1 \subseteq \mathfrak{p}_1 + \mathfrak{p}_2$ and $\mathfrak{q}_2 \subseteq \mathfrak{p}_1 + \mathfrak{p}_2$; in turn, it follows by the assumptions on $\mathfrak{p}_1$ and $\mathfrak{p}_2$ that it suffices to show that $\mathfrak{p}_1 + \mathfrak{p}_2 \not\subseteq \mathfrak{q}_1$ and $\mathfrak{p}_1 + \mathfrak{p}_2 \not\subseteq \mathfrak{q}_2$. Assume for contradiction that $\mathfrak{p}_1 + \mathfrak{p}_2 \subseteq \mathfrak{q}_1$; then $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are incomparable prime ideals in $\mathfrak{p}_2^{\uparrow}$, contradicting that $\mathfrak{p}_2^{\uparrow}$ is a chain, therefore $\mathfrak{q}_1 \subseteq \mathfrak{p}_1 + \mathfrak{p}_2$. The proof of $\mathfrak{q}_2 \subseteq \mathfrak{p}_1 + \mathfrak{p}_2$ is analogous. □

Lemma 4.3.4. *Let A be a local real closed ring and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \mathrm{Spec}(A)$ $(n \in \mathbb{N}^{\geq 2})$ be pairwise incomparable under subset inclusion. For all $i \in [n]$ there exists $j \in [n] \setminus \{i\}$ such that $\sum_{k=1}^{n} \mathfrak{p}_k = \mathfrak{p}_i + \mathfrak{p}_j$.*

Proof. The proof is by induction on $n \in \mathbb{N}^{\geq 2}$. The base case is clear, so assume that the statement holds for some $n \in \mathbb{N}^{>2}$ and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1} \in \mathrm{Spec}(A)$ be pairwise incomparable under subset inclusion. Let $i \in [n+1]$ be arbitrary and pick $j_0 \in [n+1] \setminus \{i\}$. By inductive hypothesis, there exists $j \in [n+1] \setminus \{i, j_0\}$ such that $\sum_{k \in [n+1] \setminus \{i\}} \mathfrak{p}_k = \mathfrak{p}_{j_0} + \mathfrak{p}_j$, hence $\sum_{k=1}^{n+1} \mathfrak{p}_k = \mathfrak{p}_i + \mathfrak{p}_{j_0} + \mathfrak{p}_j$; since A is a local real closed ring, either $\mathfrak{p}_i + \mathfrak{p}_j \subseteq \mathfrak{p}_i + \mathfrak{p}_{j_0}$ or $\mathfrak{p}_i + \mathfrak{p}_{j_0} \subseteq \mathfrak{p}_i + \mathfrak{p}_j$ by items (II) (iii) and (II) (iv) (a) in Theorem 2.3.2, hence either $\sum_{k=1}^{n+1} \mathfrak{p}_k = \mathfrak{p}_i + \mathfrak{p}_{j_0}$ or $\sum_{k=1}^{n+1} \mathfrak{p}_k = \mathfrak{p}_i + \mathfrak{p}_j$, as required. □

Proposition 4.3.5. *Let A be a local real closed ring of rank $n \in \mathbb{N}^{\geq 2}$. The following are equivalent:*

(i) *$\mathfrak{m}_A$ is a branching ideal.*

(ii) *For all* $\mathfrak{q}_1 \in \operatorname{Spec}(A)$ *there exists* $\mathfrak{q}_2 \in \operatorname{Spec}(A) \setminus \{\mathfrak{q}_1\}$ *such that* $\mathfrak{q}_1, \mathfrak{q}_2 \subsetneq \mathfrak{m}_A$ *and* $\mathfrak{m}_A = \mathfrak{q}_1 + \mathfrak{q}_2$.

(iii) *There exist distinct* $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}^{\min}(A)$ *such that* $\mathfrak{m}_A = \mathfrak{p}_1 + \mathfrak{p}_2$.

(iv) *For all* $\mathfrak{p}_1 \in \operatorname{Spec}^{\min}(A)$ *there exists* $\mathfrak{p}_2 \in \operatorname{Spec}^{\min}(A)\setminus\{\mathfrak{p}_1\}$ *such that* $\mathfrak{m}_A = \mathfrak{p}_1 + \mathfrak{p}_2$.

(v) *Every non-unit of* A *is a sum of two zero divisors.*

(vi) *There exists a partition* $S_1 \dot\cup S_2 = \operatorname{Spec}^{\min}(A)$ *such that* $(\bigcap_{\mathfrak{p}\in S_1} \mathfrak{p}) + (\bigcap_{\mathfrak{p}\in S_2} \mathfrak{p}) = \mathfrak{m}_A$.

(vii) *There exist* $r, s \in [n]$ *and local real closed rings* A_1 *and* A_2 *with isomorphic residue field* K *such that* $r + s = n$, $\operatorname{rk}(A_1) = r$, $\operatorname{rk}(A_2) = s$, *and* $A \cong A_1 \times_K A_2$.

Proof. Since A is local of rank $n \in \mathbb{N}^{\geq 2}$, $\operatorname{Spec}^{\min}(A) := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

(i) $\Leftrightarrow$ (ii). One implication is clear, so suppose that (i) holds and assume for contradiction that there exists $\mathfrak{q} \in \operatorname{Spec}(A)$ such that $\mathfrak{q} + \mathfrak{q}' \subsetneq \mathfrak{m}_A$ for all $\mathfrak{q}' \in \operatorname{Spec}(A)$ with $\mathfrak{q} \neq \mathfrak{q}'$. By (i), there exist distinct $\mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Spec}(A)$ such that $\mathfrak{q}_1, \mathfrak{q}_2 \subsetneq \mathfrak{m}_A$ and $\mathfrak{m}_A = \mathfrak{q}_1 + \mathfrak{q}_2$; in particular, $\mathfrak{q} \neq \mathfrak{q}_1$ and $\mathfrak{q} \neq \mathfrak{q}_2$. Since A is a local real closed ring, either $\mathfrak{q} + \mathfrak{q}_1 \subseteq \mathfrak{q} + \mathfrak{q}_2$ or $\mathfrak{q} + \mathfrak{q}_2 \subseteq \mathfrak{q} + \mathfrak{q}_1$, and assuming without loss of generality the former, it follows that $\mathfrak{m}_A = \mathfrak{q}_1 + \mathfrak{q}_2 = \mathfrak{q} + (\mathfrak{q}_1 + \mathfrak{q}_2) \subseteq \mathfrak{q} + \mathfrak{q}_2 \subsetneq \mathfrak{m}_A$, giving the required contradiction.

(i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv). Clear by Lemma 4.3.3 noting that if $\mathfrak{q}, \mathfrak{q}' \in \operatorname{Spec}(A)$ are distinct prime ideals such that $\mathfrak{q}, \mathfrak{q}' \subsetneq \mathfrak{m}_A$ and $\mathfrak{q} + \mathfrak{q}' = \mathfrak{m}_A$, then $\mathfrak{q}$ and $\mathfrak{q}'$ are incomparable under subset inclusion.

(iii) $\Rightarrow$ (v). Clear since $\mathfrak{m}_A$ consists exactly of the non-units of A and since the set of zero divisors of A is exactly $\bigcup_{i=1}^n \mathfrak{p}_i$ as A is reduced.

(v) $\Rightarrow$ (iii). Since A is a reduced local ring, (v) is equivalent to the statement that for all $\varepsilon \in \mathfrak{m}_A$ there exist $i, j \in [n]$ such that $\varepsilon = b_i + b_j$ for some $b_i \in \mathfrak{p}_i$ and $b_j \in \mathfrak{p}_j$, i.e., $\mathfrak{m}_A \subseteq \bigcup_{i,j\in[n]}(\mathfrak{p}_i + \mathfrak{p}_j)$; by Lemma 4.3.4, there exist $i', j' \in [n]$ with $i' \neq j'$ such that $\sum_{i=1}^n \mathfrak{p}_i = \mathfrak{p}_{i'} + \mathfrak{p}_{j'}$, therefore

$$\bigcup_{i,j\in[n]} (\mathfrak{p}_i + \mathfrak{p}_j) \subseteq \sum_{i=1}^{n} \mathfrak{p}_i = \mathfrak{p}_{i'} + \mathfrak{p}_{j'} \subseteq \mathfrak{m}_A \subseteq \bigcup_{i,j\in[n]} (\mathfrak{p}_i + \mathfrak{p}_j)$$

implies $\mathfrak{m}_A = \mathfrak{p}_{i'} + \mathfrak{p}_{j'}$, as required (note that $\mathfrak{p}_{i'}, \mathfrak{p}_{j'} \subsetneq \mathfrak{m}_A$ since $\operatorname{rk}(A) \geq 2$).

(iv) ⇒ (vi). Pick any $\mathfrak{p}_i \in \operatorname{Spec}^{\min}(A)$ and define $S_1 := \{\mathfrak{p} \in \operatorname{Spec}^{\min}(A) \mid \mathfrak{p}_i + \mathfrak{p} = \mathfrak{m}_A\}$ and $S_2 := \{\mathfrak{p} \in \operatorname{Spec}^{\min}(A) \mid \mathfrak{p}_i + \mathfrak{p} \neq \mathfrak{m}_A\}$. Note that $\mathfrak{p}_i \in S_2$, and also $S_1 \neq \varnothing$ by (iv), therefore $S_1 \dot\cup S_2 = \operatorname{Spec}^{\min}(A)$ is a partition.

Claim. If $\mathfrak{p} \in S_1$ and $\mathfrak{q} \in S_2$, then $\mathfrak{p} + \mathfrak{q} = \mathfrak{m}_A$.

Proof of Claim. Since $\mathfrak{q} \subseteq \mathfrak{p} + \mathfrak{q}, \mathfrak{p}_i + \mathfrak{q}$, either $\mathfrak{p} + \mathfrak{q} \subseteq \mathfrak{p}_i + \mathfrak{q}$ or $\mathfrak{p}_i + \mathfrak{q} \subseteq \mathfrak{p} + \mathfrak{q}$; in the former case it follows that $\mathfrak{m}_A = \mathfrak{p}_i + (\mathfrak{p} + \mathfrak{q}) \subseteq \mathfrak{p}_i + \mathfrak{q}$, a contradiction to $\mathfrak{q} \in S_2$, therefore $\mathfrak{p}_i + \mathfrak{q} \subseteq \mathfrak{p} + \mathfrak{q}$, hence $\mathfrak{p}_i, \mathfrak{p} \subseteq \mathfrak{p} + \mathfrak{q}$, and thus $\mathfrak{m}_A = \mathfrak{p}_i + \mathfrak{p} \subseteq \mathfrak{p} + \mathfrak{q}$ from which the claim follows. $\square_{\text{Claim}}$

Therefore

$$\left(\bigcap_{\mathfrak{p} \in S_1} \mathfrak{p}\right) + \left(\bigcap_{\mathfrak{p} \in S_2} \mathfrak{p}\right) \overset{(1)}{=} \bigcap_{\mathfrak{p} \in S_1, \mathfrak{q} \in S_2} (\mathfrak{p} + \mathfrak{q}) \overset{(2)}{=} \mathfrak{m}_A,$$

where (1) follows from Theorem 2.3.2 (II) (iv) (b) and (2) follows from the claim above.

(vi) ⇒ (vii). Let $S_1 \dot\cup S_2 = \operatorname{Spec}^{\min}(A)$ be a partition such that $(\bigcap_{\mathfrak{p} \in S_1} \mathfrak{p}) + (\bigcap_{\mathfrak{p} \in S_2} \mathfrak{p}) = \mathfrak{m}_A$. Set $r := |S_1|$, $s := |S_2|$, $A_1 := A/\bigcap_{\mathfrak{p} \in S_1} \mathfrak{p}$, and $A_2 := A/\bigcap_{\mathfrak{p} \in S_2} \mathfrak{p}$; then A_1 and A_2 are local real closed rings of ranks r and s (respectively) with isomorphic residue field $A/\mathfrak{m}_A =: K$ such that $r + s = n$ and $A \cong A_1 \times_K A_2$, where the latter statement follows from Lemma 4.2.14.

(vii) ⇒ (i). Clear by the description of the Zariski spectrum of the fibre product $A_1 \times_K A_2$ as $\operatorname{Spec}(A_1 \times_K A_2) \cong \operatorname{Spec}(A_1) \amalg_{\operatorname{Spec}(K)} \operatorname{Spec}(A_2)$ and noting that neither A_1 nor A_2 are fields, see [DST19, Section 12.5.7] (for an algebraic proof, pick $\mathfrak{r}_i \in \operatorname{Spec}(A_i) \setminus \{\mathfrak{m}_{A_i}\}$ and set $\mathfrak{q}_i := \ker(A_1 \times_K A_2 \twoheadrightarrow A_i/\mathfrak{r}_i) \in \operatorname{Spec}(A_1 \times_K A_2)$; then $\mathfrak{q}_1 = \mathfrak{r}_1 \times \mathfrak{m}_{A_2}$, $\mathfrak{q}_2 = \mathfrak{m}_{A_1} \times \mathfrak{r}_2$, and $\mathfrak{m}_A = \mathfrak{m}_{A_1} \times \mathfrak{m}_{A_2}$ as subsets of $A_1 \times_K A_2 \subseteq A_1 \times A_2$, hence $\mathfrak{m}_A = \mathfrak{q}_1 + \mathfrak{q}_2$ is a branching ideal). $\square$

Remark 4.3.6. Let A be a real closed ring of finite rank and $\mathfrak{q} \in \operatorname{Spec}(A)$; clearly $\mathfrak{q}$ is a branching ideal if and only if $\mathfrak{m}_{A_\mathfrak{q}} = \mathfrak{q}A_\mathfrak{q}$ is a branching ideal of $A_\mathfrak{q}$, therefore Proposition 4.3.5 can be applied to give equivalent characterizations for an arbitrary prime ideal of A to be a branching ideal; in particular, $\mathfrak{q} \in \operatorname{Spec}(A)$ is a branching ideal if and only if there exist distinct $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}^{\min}(A)$ such that $\mathfrak{q} = \mathfrak{p}_1 + \mathfrak{p}_2$ (this also follows immediately from Lemma 4.3.3).

Remark 4.3.7. A local real closed ring A of rank $n \in \mathbb{N}$ can have at most $n - 1$ branching ideals; in particular, the set of branching ideals of A is finite. If $n = 1$, then A is a domain (Lemma 4.2.9 (III) (i)), therefore A has no branching ideals by Remark

4.3.2 (ii); the statement for $n \in \mathbb{N}^{\geq 2}$ follows by induction using Lemma 4.2.14 and Theorem 2.3.2 (II) (iii).

Lemma 4.3.8. *Let A and B be local real closed rings of rank $n \in \mathbb{N}^{\geq 2}$ and $f : A \hookrightarrow B$ be an injective ring homomorphism. If $\mathfrak{m}_A$ is a branching ideal, then f is a local map.*

Proof. By assumption and by the implication (i) $\Rightarrow$ (ii) in Proposition 4.3.5, there exist two distinct minimal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq A$ such that $\mathfrak{m}_A = \mathfrak{p}_1 + \mathfrak{p}_2$; by Corollary 4.2.11, there exist $\mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Spec}^{\min}(B)$ such that $f^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$ and $f^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$, therefore

$$\mathfrak{m}_A = \mathfrak{p}_1 + \mathfrak{p}_2 = f^{-1}(\mathfrak{q}_1) + f^{-1}(\mathfrak{q}_2) \subseteq f^{-1}(\mathfrak{q}_1 + \mathfrak{q}_2) \subseteq f^{-1}(\mathfrak{m}_B),$$

hence $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$ by maximality of $\mathfrak{m}_A$. □

4.4 Rings of type $(n, 1)$ and of type $(n, 2)$

The starting point of this subsection is a structure theorem for local real closed SV-rings of finite rank (Theorem 4.4.2) which is directly deduced from the structure theorem for reduced local SV-rings of finite rank (Theorem 4.2.2); first, a lemma:

Lemma 4.4.1. *Let A and B be real closed rings. Suppose that there exist ideals $I \subsetneq A$ and $J \subsetneq B$ such that $A/I \cong B/J$. The fibre product $C := A \times_{B/J} B$ is real closed if and only if I and J are radical ideals of A and B, respectively; in particular, if either A or B are real closed valuation rings, then C is real closed if and only if both I and J are prime ideals of A and B, respectively.*

Proof. If I and J are radical ideals, then C is real closed by items (I) and (II) (i) in Theorem 2.3.2. Conversely, suppose that C is real closed and let $p_A : C \twoheadrightarrow A$ and $p_B : C \twoheadrightarrow B$ be the canonical surjections. Since both A and B are real closed, $\ker(p_A)$ and $\ker(p_B)$ are radical ideals of C (Theorem 2.3.2 (II) (i)), and since C is real closed, $\ker(p_A) \oplus \ker(p_B)$ is also a radical ideal (Theorem 2.3.2 (II) (iv)), therefore I and J are radical ideals by Remark 4.2.17; the last statement of the lemma follows from $B/J \cong A/I$ and from the fact that radical ideals in valuation rings are prime ideals. □

Theorem 4.4.2. *Let $n \in \mathbb{N}^{\geq 2}$ and A be a ring which is not a field. The following are equivalent:*

(i) *A is a local real closed SV-ring of rank at most n.*

(ii) *A is a local real closed ring, and there exist non-trivial real closed valuation rings $A_1, \dots, A_n$ and an injective ring homomorphism $\varepsilon : A \hookrightarrow \prod_{i=1}^n A_i$ such that $\varepsilon(A)$ is a subdirect product of $\{A_1, \dots, A_n\}$.*

(iii) *There exist non-trivial real closed valuation rings $A_1, \dots, A_n$ and an injective ring homomorphism $\varepsilon : A \hookrightarrow \prod_{i=1}^n A_i$, such that $(\pi_1^n \circ \varepsilon)(A) = A_1$ (Notation 4.2.19), and for all $j \in [n-1]$ there exist prime ideals $\mathfrak{i}_j \subsetneq (\pi_j^n \circ \varepsilon)(A)$ and $\mathfrak{j}_{j+1} \subsetneq A_{j+1}$ such that $(\pi_j^n \circ \varepsilon)(A)/\mathfrak{i}_j \cong A_{j+1}/\mathfrak{j}_{j+1}$ and*

$$(\pi_{j+1}^n \circ \varepsilon)(A) = (\pi_j^n \circ \varepsilon)(A) \times_{A_{j+1}/\mathfrak{j}_{j+1}} A_{j+1};$$

in particular,

$$A \cong (\pi_n^n \circ \varepsilon)(A) = (((A_1 \times_{A_2/\mathfrak{j}_2} A_2) \times_{A_3/\mathfrak{j}_3} A_3) \cdots \times_{A_{n-1}/\mathfrak{j}_{n-1}} A_{n-1}) \times_{A_n/\mathfrak{j}_n} A_n.$$

(iv) *There exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subseteq A$ such that $\bigcap_{i=1}^n \mathfrak{p}_i = (0)$, $\mathfrak{q}_j := (\bigcap_{i=1}^{j-1} \mathfrak{p}_i) + \mathfrak{p}_j \in \operatorname{Spec}(A)$ for all $j \in \{2, \dots, n\}$, $A/\mathfrak{p}_i$ is a non-trivial real closed valuation ring for all $i \in [n]$, and the canonical ring homomorphism $A \longrightarrow \prod_{i=1}^n A/\mathfrak{p}_i$ given by $a \mapsto (a/\mathfrak{p}_1, \dots, a/\mathfrak{p}_n)$ restricts to an isomorphism*

$$A \cong \left(\left(\left(\frac{A}{\mathfrak{p}_1} \times_{\frac{A}{\mathfrak{q}_2}} \frac{A}{\mathfrak{p}_2}\right) \times_{\frac{A}{\mathfrak{q}_3}} \frac{A}{\mathfrak{p}_3}\right) \cdots \times_{\frac{A}{\mathfrak{q}_{n-1}}} \frac{A}{\mathfrak{p}_{n-1}}\right) \times_{\frac{A}{\mathfrak{q}_n}} \frac{A}{\mathfrak{p}_n}.$$

Moreover, if any of the items (i) - (iv) *holds and* $\operatorname{rk}(A) = n$, *then the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subseteq A$ in item* (iv) *are pairwise incomparable under subset inclusion and* $\operatorname{Spec}^{\min}(A) = \{\mathfrak{p}_i \mid i \in [n]\}$.

Proof. Note first that each of the items (i) - (iv) in the theorem implies the corresponding item in Theorem 4.2.22.

(i) ⇒ (ii). By the implication (i) ⇒ (ii) in Theorem 4.2.22, it suffices to show that the non-trivial valuation rings $A_1, \dots, A_n$ in item (ii) of Theorem 4.2.22 are real closed. Since $\pi_i \circ \varepsilon : A \twoheadrightarrow A_i$ is surjective and A_i is a domain for each $i \in [n]$, $\ker(\pi_i \circ \varepsilon)$ is a prime ideal of A, hence radical, therefore $A_i \cong A/\ker(\pi_i \circ \varepsilon)$ is real closed since A is real closed (Theorem 2.3.2 (II) (i)).

(ii) $\Rightarrow$ (iii). By the implication (ii) $\Rightarrow$ (iii) in Theorem 4.2.22, it suffices to show that the ideals $I_j \subsetneq (\pi_j^n \circ \varepsilon)(A)$ and $J_{j+1} \subsetneq A_{j+1}$ in item (iii) of Theorem 4.2.22 are prime for all $j \in [n-1]$. Pick $j \in [n-1]$; since $(\pi_{j+1}^n \circ \varepsilon)(A)$ is a subring of $\prod_{i=1}^{j+1} A_i$, $(\pi_{j+1}^n \circ \varepsilon)(A)$ is reduced, and thus $\ker(\pi_{j+1}^n \circ \varepsilon)$ is a radical ideal of A, and since A is real closed, it follows that $(\pi_{j+1}^n \circ \varepsilon)(A)$ is also real closed for all $j \in \{1, \ldots, n-1\}$ by Theorem 2.3.2 (II) (i). Since $\pi_1^n(A) = \pi_1(A) = A_1$ is real closed, it now follows from $(\pi_{j+1}^n \circ \varepsilon)(A) \cong (\pi_j^n \circ \varepsilon)(A) \times_{A_{j+1}/J_{j+1}} A_{j+1}$ and from Lemma 4.4.1 that $I_j \subsetneq \pi_j^n(A)$ and $J_{j+1} \subsetneq A_{j+1}$ are prime ideals for all $j \in \{1, \ldots, n-1\}$.

(iii) $\Rightarrow$ (iv). By the implication (iii) $\Rightarrow$ (iv) in Theorem 4.2.22, it suffices to show that the non-trivial valuation ring $A/\mathfrak{p}_i$ is real closed for all $i \in [n]$ and that $G_j \in \operatorname{Spec}(A)$ for all $j \in \{2, \ldots, n\}$, where $\mathfrak{p}_i$ and G_j are the ideals of A in item (iv) of Theorem 4.2.22. By (iii) and Lemma 4.4.1, A is real closed, therefore $A/\mathfrak{p}_i$ is real closed by Theorem 2.3.2 (II) (i). Since A is real closed and the intersection of radical ideals is radical, $G_j = (\bigcap_{i=1}^{j-1} \mathfrak{p}_i) + \mathfrak{p}_j$ is a radical ideal of A by Theorem 2.3.2 (II) (iv), hence $G_j/\mathfrak{p}_j$ is a radical ideal of the valuation ring $A/\mathfrak{p}_j$, therefore it is prime, from which $G_j \in \operatorname{Spec}(A)$ follows.

(iv) $\Rightarrow$ (i). By the implication of (iv) $\Rightarrow$ (i) in Theorem 4.2.22, it suffices to argue that A is real closed; but this follows from Lemma 4.4.1 together with the fact that each $A/\mathfrak{p}_i$ is real closed and that each of the ideals $\mathfrak{q}_j$ are prime.

Finally, if any of the items (i) - (iv) holds and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subseteq A$ are the prime ideals in item (iv), then A is a local ring and a subdirect product of the domains $\{A/\mathfrak{p}_1, \ldots, A/\mathfrak{p}_n\}$, therefore if $\operatorname{rk}(A) = n$, then $\operatorname{Spec}^{\min}(A) = \{\mathfrak{p}_i \mid i \in [n]\}$ follows by Lemma 4.2.13 (II). □

Remark 4.4.3. Let A be a local real closed SV-ring of rank $n \in \mathbb{N}^{\geq 2}$ and write $\operatorname{Spec}^{\min}(A) = \{\mathfrak{p}_i \mid i \in [n]\}$; the ideals $\mathfrak{q}_j := (\bigcap_{i=1}^{j-1} \mathfrak{p}_i) + \mathfrak{p}_j \in \operatorname{Spec}(A)$ ($j \in [n-1]$) in item (iv) of Theorem 4.4.2 are exactly the branching ideals of A (note that it could be the case that $\mathfrak{q}_i = \mathfrak{q}_j$ for $i \neq j$). Pick $j \in [n-1]$; then $\mathfrak{q}_j = (\bigcap_{i=1}^{j-1} \mathfrak{p}_i) + \mathfrak{p}_j = \bigcap_{i=1}^{j-1}(\mathfrak{p}_i + \mathfrak{p}_j) = \mathfrak{p}_k + \mathfrak{p}_j$ for some $k \in [j-1]$ by items (II) (iii), (II) (iv) (a), and (II) (iv) (b) in Theorem 2.3.2, therefore $\mathfrak{q}_j$ is a branching ideal. Conversely, suppose that $\mathfrak{q} \in \operatorname{Spec}(A)$ is a branching ideal; By Remark 4.3.6 there exist distinct $i, j \in [n]$ such that $i \neq j$ and $\mathfrak{q} = \mathfrak{p}_i + \mathfrak{p}_j$. Let $i \in [n]$ be minimal with the property that there exists $j_0 \in [n]$ such that $i < j_0$ and $\mathfrak{p}_i + \mathfrak{p}_{j_0} = \mathfrak{q}$, and let $j \in [n]$ be minimal with the property that $i < j$ and $\mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{q}$. Assume for contradiction that there exists $k \in [j-1]$ such that $\mathfrak{p}_k + \mathfrak{p}_j \subsetneq \mathfrak{p}_i + \mathfrak{p}_j$; note in particular that $\mathfrak{p}_i + \mathfrak{p}_k \subseteq \mathfrak{p}_i + \mathfrak{p}_j$. Then $\mathfrak{p}_i + \mathfrak{p}_k \not\subseteq \mathfrak{p}_j + \mathfrak{p}_k$ as otherwise $\mathfrak{p}_i \subseteq \mathfrak{p}_i + \mathfrak{p}_k \subseteq \mathfrak{p}_j + \mathfrak{p}_k$ implies $\mathfrak{p}_i + \mathfrak{p}_j \subseteq \mathfrak{p}_j + \mathfrak{p}_k \subsetneq \mathfrak{p}_i + \mathfrak{p}_j$, therefore $\mathfrak{p}_j + \mathfrak{p}_k \subseteq \mathfrak{p}_i + \mathfrak{p}_k$, and thus $\mathfrak{p}_i + \mathfrak{p}_j \subseteq \mathfrak{p}_i + \mathfrak{p}_k \subseteq \mathfrak{p}_i + \mathfrak{p}_j$, hence $\mathfrak{p}_i + \mathfrak{p}_k = \mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{q}$, a contradiction to minimality of j. Therefore $\mathfrak{p}_i + \mathfrak{p}_j \subseteq \mathfrak{p}_k + \mathfrak{p}_j$ for all $k \in [j-1]$, and thus $\mathfrak{q}_j = (\bigcap_{k\in[j-1]} \mathfrak{p}_k) + \mathfrak{p}_j = \bigcap_{k\in[j-1]}(\mathfrak{p}_k + \mathfrak{p}_j) = \mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{q}$, as required.

Remark 4.4.4. It is clear from the proof of Theorem 4.4.2 that a similar structure theorem holds for local real closed rings of finite rank: just replace every occurrence of "non-trivial real closed valuation ring" by "non-trivial real closed domain".

Remark 4.4.5. Any real closed ring has *bounded inversion* ([SM99, Example 2.11, Proposition 12.4. (b)]), therefore a real closed domain is a real closed valuation ring if and only if it is 1-*convex* by [Lar10, Lemma 2.2]; in particular, local real closed SV-rings of finite rank are particular examples of Larson's *finitely* 1-*convex* f-*rings*, see [Lar11].

Let A be a local real closed SV-ring of rank $n \in \mathbb{N}^{\geq 2}$; by the implication (i) $\Rightarrow$ (iv) in Theorem 4.4.2, A is isomorphic to a finite iterated fibre product of non-trivial real closed valuation rings $A/\mathfrak{p}_i$ along surjective homomorphisms $A/\mathfrak{p}_i \twoheadrightarrow A/\mathfrak{q}_j$ onto real closed valuation rings $A/\mathfrak{q}_j$, and the simplest such rings are the ones for which $A/\mathfrak{q}_i \cong A/\mathfrak{q}_j$ for all $i, j \in [n]$, i.e., local real closed SV-rings of rank n with exactly one branching ideal (Remark 4.4.3). The focus of the remaining part of this section is on rings of this latter class.

Notation 4.4.6. Let $\{A_i\}_{i\in I}$ be a non-empty family of rings such that there exists a ring B and surjective ring homomorphisms $f_i : A_i \twoheadrightarrow B$ for all $i \in I$. Let

$$\prod\nolimits_{B,i\in I} A_i := \left\{ (a_i)_{i\in I} \in \prod_{i\in I} A_i \mid f_i(a_i) = f_j(a_j) \text{ for all } i, j \in I \right\}$$

be the *I-fold fibre product of $\{A_i\}_{i\in I}$ over B (along $\{f_i\}_{i\in I}$)*. If there exists a ring A such that $A_i = A$ for all $i \in I$, then set $\prod_B^I A := \prod_{B,i\in I} A_i$; if moreover $I = [n]$ for some $n \in \mathbb{N}$, then set $\prod_B^n A := \prod_B^{[n]} A$.

Lemma 4.4.7. *Let A be a local real closed SV-ring of rank $n \in \mathbb{N}^{\geq 2}$ and write* $\operatorname{Spec}^{\min}(A) = \{\mathfrak{p}_i \mid i \in [n]\}$. *The following are equivalent:*

(i) *A has exactly one branching ideal.*

(ii) $\mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{p}_k + \mathfrak{p}_\ell$ *for all* $i, j, k, \ell \in [n]$ *such that* $i \neq j$ *and* $k \neq \ell$.

(iii) *There exist non-trivial real closed valuation rings* $A_1, \dots, A_n$ *and surjective ring homomorphisms* $f_i : A_i \twoheadrightarrow C$ $(i \in [n])$ *onto a real closed valuation ring* C *such that* $\ker(f_i) \neq (0)$ *for all* $i \in [n]$ *and* $A \cong \prod_{C, i\in[n]} A_i$.

In particular, if any of the conditions (i) - (iii) *holds and* $\mathfrak{b}_A \in \operatorname{Spec}(A)$ *is the unique branching ideal of* A*, then*

(a) $\mathfrak{b}_A = \mathfrak{p}_i + \mathfrak{p}_j$ *for all* $i, j \in [n]$ *such that* $i \neq j$*, and*

(b) *the canonical embedding* $A \hookrightarrow \prod_{i\in[n]} A/\mathfrak{p}_i$ *given by* $a \mapsto (a/\mathfrak{p}_i)_{i\in[n]}$ *corestricts to an isomorphism* $A \cong \prod_{A/\mathfrak{b}_A, i\in[n]} A/\mathfrak{p}_i$

Proof. (i) $\Rightarrow$ (ii). Clear by Remark 4.3.6.

(ii) $\Rightarrow$ (iii). Since A is local of rank n, $\operatorname{Spec}^{\min}(A) := \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Define $A_i := A/\mathfrak{p}_i$ and $\mathfrak{q}_j := (\bigcap_{i=1}^{j-1} \mathfrak{p}_i) + \mathfrak{p}_j$ for all $j \in \{2, \dots, n\}$. If $j \in \{2, \dots, n\}$, then $\mathfrak{q}_j = \bigcap_{i=1}^{j-1}(\mathfrak{p}_i + \mathfrak{p}_j)$ by Theorem 2.3.2 (II) (iv) (b), and since $\mathfrak{p}_i + \mathfrak{p}_j \in \operatorname{Spec}(A)$ for all $i \in [j-1]$ by Theorem 2.3.2 (II) (iv) (a), there exists $j' \in [j-1]$ such that $\mathfrak{q}_j = \bigcap_{i=1}^{j-1}(\mathfrak{p}_i + \mathfrak{p}_j) = \mathfrak{p}_{j'} + \mathfrak{p}_j$ by Theorem 2.3.2 (II) (iii), therefore it follows by (ii) that $\mathfrak{q}_j = \mathfrak{q}_i$ for all $i, j \in \{2, \dots, n\}$. Define $\mathfrak{q} := \mathfrak{q}_j$ for some (equivalently, all) $j \in \{2, \dots, n\}$; then $A \cong \prod_{A/\mathfrak{q}, i\in[n]} A/\mathfrak{p}_i$ by the implication (i) $\Rightarrow$ (iv) and the moreover part in Theorem 4.4.2, from which (iii) follows.

(iii) $\Rightarrow$ (i). $\operatorname{Spec}(C)$ is a closed subspace of $\operatorname{Spec}(A)$ with unique generic point $\mathfrak{q} = \ker(A \twoheadrightarrow C)$, therefore follows from the description in [DST19, Section 12.5.7] of $\operatorname{Spec}(A)$ that the unique branching ideal of A is $\mathfrak{q}$.

The last statement in the lemma follows from the proof of the equivalences (i) - (iii). □

Definition 4.4.8. Let A be a ring and $n \in \mathbb{N}^{\geq 2}$.

(i) A is of *type* $(n, 1)$ if A is a local real closed SV-ring of rank n with exactly one branching ideal $\mathfrak{b}_A$, which moreover is maximal.

(ii) A is of *type* $(n, 2)$ if A is a local real closed SV-ring of rank n with exactly one branching ideal $\mathfrak{b}_A$, which moreover is not maximal.

Example 4.4.9. Let R be a real closed field, $X \subseteq R^m$ be a semi-algebraic curve, and $a \in X$ be a point with branching degree $n \in \mathbb{N}^{\geq 2}$ (see Subsection 2.3.2). It follows by the implication (iii) $\Rightarrow$ (i) in Lemma 4.4.7 and Corollary 2.3.37 that the ring of germs of continuous semi-algebraic functions $X \longrightarrow R$ at a is a ring of type $(n, 1)$.

4.4.1 Embeddings of rings of type $(n, 1)$ and of type $(n, 2)$

This section concludes with embedding statements about rings of type $(n, 1)$ and of type $(n, 2)$ which have a key impact on the model theory of these rings; in particular, Lemmas 4.4.21 and 4.4.22 (which can be thought of as a higher-rank version of Proposition 2.3.9) are essential for the model completeness proofs in Section 4.5. Start with the following and almost trivial:

Lemma 4.4.10. *Let A be a local real closed SV-ring of rank $n \in \mathbb{N}^{\geq 2}$. There exists a ring A' of type $(n, 1)$ and a local embedding $A \hookrightarrow A'$.*

Proof. Set $\operatorname{Spec}^{\min}(A) = \{\mathfrak{p}_i \mid i \in [n]\}$. The canonical map $A \hookrightarrow \prod_{i\in[n]} A/\mathfrak{p}_i$ given by $a \mapsto (a/\mathfrak{p}_1, \dots, a/\mathfrak{p}_n)$ is an embedding, and each $A/\mathfrak{p}_i$ is a non-trivial real closed valuation ring with residue field $A/\mathfrak{m}_A$, from which follows that $A' := \prod_{A/\mathfrak{m}_A, i\in[n]} A/\mathfrak{p}_i$ is a ring of type $(n, 1)$ and the embedding $A \hookrightarrow \prod_{i\in[n]} A_i$ corestricts to a local embedding $A \hookrightarrow A'$. □

In fact, Lemma 4.4.10 still holds even if A is not an SV-ring; first, a lemma:

Lemma 4.4.11. *Let A be a real closed domain.*

(i) *$A = \operatorname{qf}(A)$ if and only if A is cofinal in $\operatorname{qf}(A)$.*

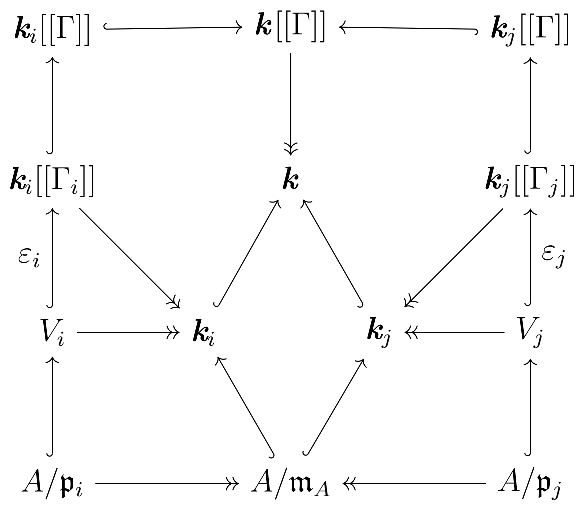
(ii) *Suppose that A is non-trivial, i.e., $A \neq \operatorname{qf}(A)$. The convex hull V_A of A in $\operatorname{qf}(A)$ is a non-trivial real closed valuation ring such that $A \cap \mathfrak{m}_{V_A} = \mathfrak{m}_A$.*

Proof. (i). Suppose that A is cofinal in $\operatorname{qf}(A)$ and pick $a \in A^{>0}$. Then there exists $b \in A$ such that $0 < a^{-1} < b$, therefore $0 < 1 < ab$ and thus there exists $c \in A$ such that $abc = 1^2 = 1$ (Definition 2.3.1 (iii)), from which $a^{-1} = bc \in A$ follows.

(ii). V_A is a non-trivial real closed valuation ring because it is a proper convex subring of $\operatorname{qf}(A)$ by item (i). Clearly $A \cap \mathfrak{m}_{V_A} \subseteq \mathfrak{m}_A$. Conversely, pick $a \in \mathfrak{m}_A$ such that $a > 0$ and assume for contradiction that $a \notin \mathfrak{m}_{V_A}$; then $a^{-1} \in V_A$, therefore there exists $b \in A$ such that $0 < a^{-1} < b$, and thus arguing as in the proof of item (i) it follows that $a^{-1} \in A$, yielding the required contradiction. $\square$

Proposition 4.4.12. *Let A be a local real closed ring of rank $n \in \mathbb{N}^{\geq 2}$. There exists a ring B of type $(n, 1)$ and a local embedding $A \hookrightarrow B$.*

Proof. Set $\operatorname{Spec}^{\min}(A) = \{\mathfrak{p}_i \mid i \in [n]\}$; each $A/\mathfrak{p}_i$ is a non-trivial real closed domain with residue field $A/\mathfrak{m}_A$, therefore the canonical embedding $A \hookrightarrow \prod_{i\in[n]} A/\mathfrak{p}_i$ given by $a \mapsto (a/\mathfrak{p}_i)_{i\in[n]}$ corestricts to a local embedding $A \hookrightarrow \prod_{A/\mathfrak{m}_A, i\in[n]} A/\mathfrak{p}_i =: B_0$. For each $i \in [n]$, the convex hull V_i of $A/\mathfrak{p}_i$ in $\operatorname{qf}(A/\mathfrak{p}_i)$ is a non-trivial real closed valuation ring by Lemma 4.4.11 (ii). Let $\boldsymbol{k}_i := V_i/\mathfrak{m}_{V_i}$ and $\Gamma_i := \operatorname{qf}(V_i)^\times/V_i^\times$ for all $i \in [n]$; note that since $A \cap \mathfrak{m}_{V_i} = \mathfrak{m}_A$ (Lemma 4.4.11 (ii)), the residue field $A/\mathfrak{m}_A$ embeds into $\boldsymbol{k}_i$ for all $i \in [n]$. Let $\boldsymbol{k}$ be a real closed field amalgamating all the $\boldsymbol{k}_i$ over $A/\mathfrak{m}_A$, let Γ be a divisible totally ordered abelian group into which all the Γ_i embed, and let $\varepsilon_i : V_i \hookrightarrow \boldsymbol{k}_i[[\Gamma_i]]$ be a local embedding for all $i \in [n]$ (these exist by Proposition 2.3.9); all this data fits into commutative diagrams



for all $i, j \in [n]$. Commutativity of the diagrams above for all $i, j \in [n]$ implies that the resulting composite local embeddings

$$A/\mathfrak{p}_i \hookrightarrow V_i \overset{\varepsilon_i}{\hookrightarrow} \boldsymbol{k}_i[[\Gamma_i]] \hookrightarrow \boldsymbol{k}_i[[\Gamma]] \hookrightarrow \boldsymbol{k}[[\Gamma]]$$

for all $i \in [n]$ yield a local embedding $B_0 \hookrightarrow \prod_{\boldsymbol{k}}^n \boldsymbol{k}[[\Gamma]] =: B$, therefore the composite map $A \hookrightarrow B_0 \hookrightarrow B$ is a local embedding of A into the ring B of type $(n, 1)$, as required. $\square$

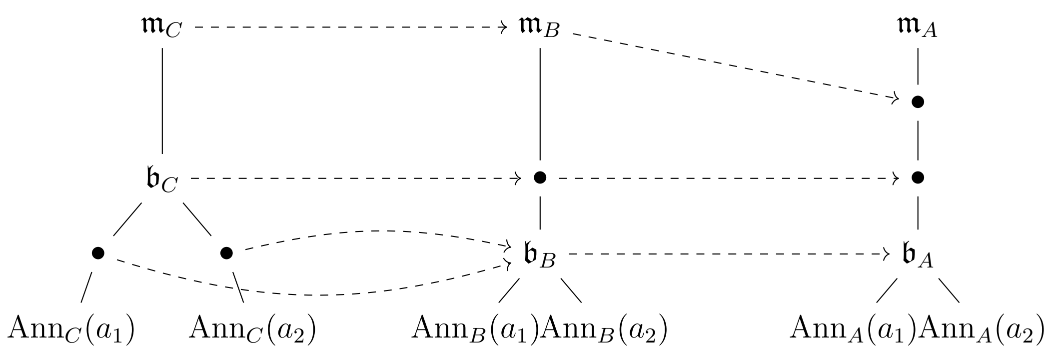
Convention 4.4.13. Let $j \in [2]$, and let A and B be rings of type (n, j) such that $A \subseteq B$; write also $\text{Spec}^{\min}(A) := \{\mathfrak{p}_i \mid i \in [n]\}$ and $\text{Spec}^{\min}(B) := \{\mathfrak{q}_i \mid i \in [n]\}$. By Corollary 4.2.11 it can be assumed that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \in [n]$, and thus by the implication (i) $\Rightarrow$ (iii) and item (b) in Lemma 4.4.7 it can be assumed that $A = \prod_{C, i\in[n]} A_i$ and $B = \prod_{D, i\in[n]} B_i$, where $A_i := A/\mathfrak{p}_i$ and $B_i := B/\mathfrak{q}_i$ for all $i \in [n]$, $C := A/\mathfrak{b}_A$, and $D := B/\mathfrak{b}_B$; in particular, the embedding $A \subseteq B$ induces embeddings $A_i \subseteq B_i$ for all $i \in [n]$.

Lemma 4.4.14. *Let A and B be rings of type $(n, 1)$. Any embedding $A \subseteq B$ is local and it induces local embeddings $A_i \subseteq B_i$ for all $i \in [n]$* (Convention 4.4.13)*.*

Proof. That any embedding $A \subseteq B$ is local follows from Lemma 4.3.8; since $A \subseteq B$ is local, so are each of the embeddings $A_i \subseteq B_i$ (Remark 4.2.12). □

Lemma 4.4.14 says that any embedding $A \subseteq B$ of rings of type $(n, 1)$ sends the unique branching ideal $\mathfrak{b}_B$ of B to the unique branching ideal $\mathfrak{b}_A$ of A, i.e., $\mathfrak{b}_B \cap A = \mathfrak{b}_A$, and since the branching ideals of these rings are exactly the maximal ideals, this means that $A \subseteq B$ is a local embedding; the next example shows that this property of arbitrary embeddings of rings of type $(n, 1)$ does not carry over to rings to type $(n, 2)$:

Example 4.4.15. Let V be a real closed valuation ring of Krull dimension 4 such that $(\text{Spec}(V), \subseteq)$ is the chain $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{q} \subsetneq \mathfrak{r} \subsetneq \mathfrak{m}_V$; define $A := V \times_{V/\mathfrak{p}} V$, $B := V_\mathfrak{r} \times_{V_\mathfrak{r}/\mathfrak{p}V_\mathfrak{r}} V_\mathfrak{r}$ and $C := V_\mathfrak{r} \times_{V_\mathfrak{r}/\mathfrak{q}V_\mathfrak{r}} V_\mathfrak{r}$. Then $A \subseteq B$ and $B \subseteq C$ are embeddings of rings of type $(2, 2)$ such that the embedding $A \subseteq B$ is not local but $\mathfrak{b}_B \cap A = \mathfrak{b}_A$, and the embedding $B \subseteq C$ is local but $\mathfrak{b}_C \cap B \supsetneq \mathfrak{b}_B$; in particular, the embedding $A \subseteq C$ is not local and $\mathfrak{b}_C \cap A \supsetneq \mathfrak{b}_A$. The diagram below represents the composite spectral map $\text{Spec}(C) \to \text{Spec}(B) \to \text{Spec}(A)$ induced by the composite embedding $A \subseteq B \subseteq C$



where $a_1, a_2 \in A$ are any non-zero orthogonal elements (Corollary 4.2.11); an analogous construction shows that for all $n \in \mathbb{N}^{\geq 2}$ there exists embeddings $A \subseteq B$ and $B \subseteq C$ of rings of type $(n, 2)$ such that $A \subseteq B$ is not a local embedding and such that $\mathfrak{b}_C \cap B \supsetneq \mathfrak{b}_B$.

Embeddings $A \subseteq B$ of rings of type $(n, 2)$ which satisfy $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ and $\mathfrak{b}_B \cap A = \mathfrak{b}_B$ play an important role in the model-theoretic analysis of this class of rings; to this end, make the following:

Definition 4.4.16. An embedding $A \subseteq B$ of rings of type $(n, 2)$ is *good* if $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ and $\mathfrak{b}_B \cap A = \mathfrak{b}_B$.

Remark 4.4.17. Let A and B be rings of type (n, j) such that $A \subseteq B$ and write $A = \prod_{C, i\in[n]} A_i$ and $B = \prod_{D, i\in[n]} B_i$ (Convention 4.4.13). If $j = 1$, then C and D are real closed fields and the embedding $A \subseteq B$ induces an embedding $C \subseteq D$ by Lemma 4.4.14; if $j = 2$, then C and D is are non-trivial real closed valuation rings and the embedding $A \subseteq B$ is good if and only if it induces a local embedding $C \subseteq D$.

Definition 4.4.18. Let $f : A \hookrightarrow B$ be an injective ring homomorphism and $\mathfrak{p} \in \mathrm{Spec}(B)$. Say that f is *sharp at* $\mathfrak{p}$ if the induced embedding $A/f^{-1}(\mathfrak{p}) \hookrightarrow B/\mathfrak{p}$ is an isomorphism.

Definition 4.4.19 (cf. Definition 6 in [Lar11])**.** Let $j \in [2]$. A ring A of type (n, j) is *homogeneous* if there exists a non-trivial real closed valuation ring V together with surjective ring homomorphism $f : V \twoheadrightarrow B$ onto a real closed valuation ring B such that $A \cong \prod_B^n V$ (Notation 4.4.6).

Example 4.4.20. Let V and W be non-trivial real closed valuation rings with isomorphic residue field $\boldsymbol{k}$. If there exists a local embedding $\varepsilon : V \hookrightarrow W$, then ε induces a local embedding $f : V \times_{\boldsymbol{k}} W \hookrightarrow W \times_{\boldsymbol{k}} W =: B$, therefore f is an embedding of a ring of type $(2, 1)$ into a homogeneous ring of type $(2, 1)$, and f is sharp at $\mathfrak{m}_B$. This construction cannot be done in general. More precisely, let $\boldsymbol{k}$ be a real closed field, Γ be a countable non-archimedean totally ordered divisible abelian group, and define $A := \boldsymbol{k}[[\Gamma]] \times_{\boldsymbol{k}} \boldsymbol{k}[[\mathbb{R}]]$; there does not exist any embedding $A \hookrightarrow \boldsymbol{k}[[\Gamma]] \times_{\boldsymbol{k}} \boldsymbol{k}[[\Gamma]]$ nor any embedding $A \hookrightarrow \boldsymbol{k}[[\mathbb{R}]] \times_{\boldsymbol{k}} \boldsymbol{k}[[\mathbb{R}]]$. Otherwise, such embeddings would induce either a local embedding $\boldsymbol{k}[[\mathbb{R}]] \hookrightarrow \boldsymbol{k}[[\Gamma]]$ or a local embedding $\boldsymbol{k}[[\Gamma]] \hookrightarrow \boldsymbol{k}[[\mathbb{R}]]$

(Lemma 4.4.14); the latter embeddings give rise to embeddings of divisible totally ordered abelian groups $\mathbb{R} \hookrightarrow \Gamma$ or $\Gamma \hookrightarrow \mathbb{R}$, which is impossible by choice of Γ.

Lemma 4.4.21. *Let A and B be rings of type $(n, 1)$ such that $A \subseteq B$. There exist rings A' and B' of type $(n, 1)$ and embeddings $\varepsilon_A : A \hookrightarrow A'$ and $\varepsilon_B : B \hookrightarrow B'$ such that $A' \subseteq B'$, ε_A is sharp at $\mathfrak{b}_{A'}$ $(= \mathfrak{m}_{A'})$, ε_B is sharp at $\mathfrak{b}_{B'}$ $(= \mathfrak{m}_{B'})$, A' and B' are homogeneous, and $\varepsilon_{B\restriction A} = \varepsilon_A$.*

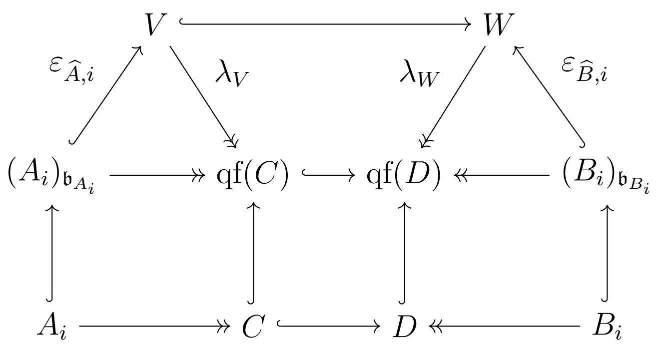
Proof. Write $A = \prod_{\boldsymbol{k}, i \in [n]} A_i$ and $B = \prod_{\boldsymbol{l}, i \in [n]} B_i$, where A_i and B_i are non-trivial real closed valuation rings with residue fields $\boldsymbol{k}$ and $\boldsymbol{l}$ (respectively) for all $i \in [n]$, see Convention 4.4.13; set also $\Gamma_i := \mathrm{qf}(A_i)^\times / A_i^\times$ and $\Delta_i := \mathrm{qf}(B_i)^\times / B_i^\times$, noting that the local embedding $A_i \subseteq B_i$ (Lemma 4.4.14) induces a local embedding $\boldsymbol{k}[[\Gamma_i]] \subseteq \boldsymbol{l}[[\Delta_i]]$ for all $i \in [n]$. By Proposition 2.3.9 there exist local embeddings $\eta_i : A_i \hookrightarrow \boldsymbol{k}[[\Gamma_i]]$ and $\delta_i : B_i \hookrightarrow \boldsymbol{l}[[\Delta_i]]$ such that $\delta_{i \restriction A_i} = \eta_i$ for all $i \in [n]$; by embedding all the divisible o-groups Δ_i into a divisible o-group Δ, it follows that the local embeddings η_i and δ_i induce local embeddings $\eta'_i : A_i \hookrightarrow \boldsymbol{k}[[\Delta]]$ and $\delta'_i : B_i \hookrightarrow \boldsymbol{l}[[\Delta]]$ such that $\delta'_{i \restriction A_i} = \eta'_i$. Define $A' := \prod_{\boldsymbol{k}}^n \boldsymbol{k}[[\Delta]]$ (Notation 4.4.6), $B' := \prod_{\boldsymbol{l}}^n \boldsymbol{l}[[\Delta]]$, ε_A to be given by $\varepsilon_A(a) := (\eta'_1(a_1), \ldots, \eta'_n(a_n))$, and ε_B to be given by $\varepsilon_B(b) := (\delta'_1(b_1), \ldots, \delta'_n(b_n))$; it is clear by construction that this choice of data satisfies the requirements in the statement of the lemma. □

Lemma 4.4.22. *Let A and B be rings of type $(n, 2)$ such that $A \subseteq B$ and such that $A \subseteq B$ is a good embedding* (Definition 4.4.16)*. There exist rings A' and B' of type $(n, 2)$, and good embeddings $\varepsilon_A : A \hookrightarrow A'$ and $\varepsilon_B : B \hookrightarrow B'$, such that $A' \subseteq B'$, the embedding $A' \subseteq B'$ is good, ε_A is sharp at $\mathfrak{b}_{A'}$, ε_B is sharp at $\mathfrak{b}_{B'}$, A' and B' are homogeneous, and $\varepsilon_{B\restriction A} = \varepsilon_A$.*

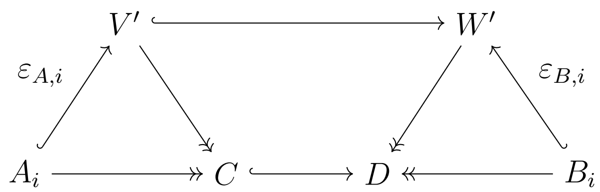
Proof. Write $A = \prod_{C, i \in [n]} A_i$ and $B = \prod_{D, i \in [n]} B_i$, where A_i and B_i are non-trivial real closed valuation rings having non-trivial real closed valuation rings C and D as homomorphic images (respectively) for all $i \in [n]$, see Convention 4.4.13; set also $\mathfrak{b}_{A_i} := \ker(A_i \twoheadrightarrow C)$ and $\mathfrak{b}_{B_i} := \ker(B_i \twoheadrightarrow D)$, noting that neither $\mathfrak{b}_{A_i}$ nor $\mathfrak{b}_{B_i}$ are the zero ideal by the implication (i) $\Rightarrow$ (iii) in Lemma 4.4.7. For each $i \in [n]$, the localization $(A_i)_{\mathfrak{b}_{A_i}}$ is a non-trivial real closed valuation ring properly containing A_i with residue field $(A_i)_{\mathfrak{b}_{A_i}} / \mathfrak{b}_{A_i} (A_i)_{\mathfrak{b}_{A_i}} = (A_i)_{\mathfrak{b}_{A_i}} / \mathfrak{b}_{A_i} = \mathrm{qf}(A_i / \mathfrak{b}_{A_i}) = \mathrm{qf}(C)$, therefore $\widehat{A} := \prod_{\mathrm{qf}(C), i \in [n]} (A_i)_{\mathfrak{b}_{A_i}}$ is a ring of type $(n, 1)$ such that $A \subseteq \widehat{A}$; similarly, $\widehat{B} :=$

$\prod_{\mathrm{qf}(D), i\in[n]}(B_i)_{\mathfrak{b}_{B_i}}$ is a ring of type $(n,1)$ such that $B \subseteq \widehat{B}$, and since $A \subseteq B$ a good embedding, it induces an embedding $\widehat{A} \subseteq \widehat{B}$ making the obvious square commute (cf. Remark 4.4.17).

By Lemma 4.4.21, there exist rings $\widehat{A}'$ and $\widehat{B}'$ of type $(n,1)$ and embeddings $\varepsilon_{\widehat{A}} : \widehat{A} \longrightarrow \widehat{A}'$ and $\varepsilon_{\widehat{B}} : \widehat{B} \longrightarrow \widehat{B}'$ such that $\widehat{A}' \subseteq \widehat{B}'$, $\varepsilon_{\widehat{A}}$ is sharp at $\mathfrak{b}_{\widehat{A}'}$ $(= \mathfrak{m}_{\widehat{A}'})$, $\varepsilon_{\widehat{B}}$ is sharp at $\mathfrak{b}_{\widehat{B}'}$ $(= \mathfrak{m}_{\widehat{B}'})$, $\widehat{A}'$ and $\widehat{B}'$ are homogeneous, and $\varepsilon_{\widehat{B}'\restriction\widehat{A}'} = \varepsilon_{\widehat{A}'}$; in particular, there exist non-trivial real closed valuation rings V and W with residue fields $\mathrm{qf}(C)$ and $\mathrm{qf}(D)$ (respectively) such that $V \subseteq W$ and making the diagram



commute for all $i \in [n]$, where λ_V and λ_W are the residue field maps and $\varepsilon_{\widehat{A},i}$ and $\varepsilon_{\widehat{B},i}$ are the embeddings induced by $\varepsilon_{\widehat{A}}$ and $\varepsilon_{\widehat{B}}$ (respectively). Define $V' := \lambda_V^{-1}(C)$ and $W' := \lambda_W^{-1}(D)$; then V' and W' are non-trivial real closed valuation rings by Lemma 2.3.10, and using the fact that $\lambda_V^{-1}(C) \cong V \times_{\mathrm{qf}(C)} C$ and $\lambda_W^{-1}(D) \cong W \times_{\mathrm{qf}(D)} D$, it follows from the universal property of the pullback that the diagram



commutes for all $i \in [n]$, where $\varepsilon_{A,i}$ and $\varepsilon_{B,i}$ are the restrictions of $\varepsilon_{\widehat{A},i}$ and $\varepsilon_{\widehat{B},i}$ to A_i and B_i (respectively), and $V' \subseteq W'$ is induced by $V \subseteq W$. Define $A' := \prod_C^n V'$, $B' := \prod_D^n W'$, $\varepsilon_A : A \longrightarrow A'$ to be given by $\varepsilon_A(a) := (\varepsilon_{A,1}(a_1), \ldots, \varepsilon_{A,n}(a_n))$, and $\varepsilon_B : B \longrightarrow B'$ to be given by $\varepsilon_B(b) := (\varepsilon_{B,1}(b_1), \ldots, \varepsilon_{B,n}(b_n))$; it is clear by construction that this choice of data satisfies the requirements in the statement of the lemma. □

4.5 Model theory

Throughout this section, set $\mathscr{L} := \mathscr{L}^{\text{ring}} = \{+, -, \cdot, 0, 1\}$.

4.5.1 The theories T_n, $T_{n,1}$, and $T_{n,2}$

Lemma 4.5.1. *There exists an $\mathscr{L}$-sentence $\varphi_{\text{rk}=n}$ such that for all reduced local rings A, $A \models \varphi_{\text{rk}=n}$ if and only if A has rank n.*

Proof. By Lemma 4.2.9 (III) (i) one may define $\varphi_{\text{rk}=n}$ to be the $\mathscr{L}$-sentence expressing the following statement about A: "there exist non-zero pairwise orthogonal elements $a_1, \dots, a_n \in A$ such that if $b \in A$ is a non-zero element distinct from all the a_i, then b is not orthogonal to some a_i". □

Lemma 4.5.2. *There exists an $\mathscr{L}$-sentence $\varphi_{\text{SV},n}$ such that for all reduced local rings A of rank n, $A \models \varphi_{\text{SV},n}$ if and only if A is an SV-ring; moreover, $\varphi_{\text{SV},n}$ can be chosen to be a universal sentence in the language $\mathscr{L}(\text{div})$, where div is a binary predicate interpreted as divisibility.*

Proof. Let $a_1, \dots, a_n \in A$ be any non-zero pairwise orthogonal elements, so that $\text{Spec}^{\text{min}}(A) = \{\text{Ann}_A(a_i) \mid i \in [n]\}$ by Lemma 4.2.9 (III) (ii). By the implication (i) ⇒ (ii) in Theorem 4.2.2, A is an SV-ring if and only if for all $b, c \in A$ and for all $i \in [n]$, $b/\text{Ann}(a_i)$ divides $c/\text{Ann}(a_i)$ or $c/\text{Ann}(a_i)$ divides $b/\text{Ann}(a_i)$ in $A/\text{Ann}_A(a_i)$; moreover,

$$
\begin{aligned}
A/\text{Ann}_A(a_i) \models \text{div}(b/\text{Ann}(a_i), c/\text{Ann}(a_i)) &\iff A \models \exists x[(bx - c)a_i = 0] \\
&\iff A \models \text{div}(ba_i, ca_i),
\end{aligned}
$$

therefore one may define $\varphi_{\text{SV},n}$ to be the $\mathscr{L}$-sentence expressing the following statement about A: "for all non-zero pairwise orthogonal $a_1, \dots, a_n \in A$ and for all $b, c \in A$, either ba_i divides ca_i or ca_i divides ba_i". □

Definition 4.5.3. Let T_n to be the $\mathscr{L}$-theory of local real closed SV-rings of rank n; explicitly, an axiomatization for T_n consists of the $\mathscr{L}$-axioms for local real closed rings ([PS, pp. 5, 9]) together with the sentences $\varphi_{\text{rk}=n}$ and $\varphi_{\text{SV},n}$ defined in Lemmas 4.5.1 and 4.5.2, respectively.

Definition 4.5.4. Let $\varphi_{\mathrm{br},n}$ to be the $\mathscr{L}$-sentence expressing the following statement about a ring A: "$\mathrm{Ann}(a_i) + \mathrm{Ann}(a_j) = \mathrm{Ann}(a_k) + \mathrm{Ann}(a_\ell)$ for all pairwise orthogonal non-zero elements $a_1, \ldots, a_n \in A$ and for all $i, j, k, \ell \in [n]$ such that $i \neq j$ and $k \neq \ell$."

Lemma 4.5.5. *The following are equivalent for all local real closed rings A of rank n:*

(i) $A \models \varphi_{\mathrm{br},n}$.

(ii) *A has exactly one branching ideal.*

Proof. By the equivalence (i) $\Leftrightarrow$ (ii) in Lemma 4.4.7 together with Lemma 4.2.9 (III) (ii). □

Definition 4.5.6. (i) Let $T_{n,1}$ be the $\mathscr{L}$-theory T_n together with $\varphi_{\mathrm{br},n}$ and the $\mathscr{L}$-sentence expressing the following statement about a ring: "every unit is a sum of two zero divisors".

(ii) Let $T_{n,2}$ be the $\mathscr{L}$-theory T_n together with $\varphi_{\mathrm{br},n}$ and the $\mathscr{L}$-sentence expressing the following statement about a ring: "there exists a unit which is not a sum of two zero divisors".

Lemma 4.5.7. *$A \models T_{n,1}$ if and only if A is a ring of type $(n, 1)$ and $A \models T_{n,2}$ if and only if A is a ring of type $(n, 2)$.*

Proof. By Lemma 4.5.5 and the equivalence (i) $\Leftrightarrow$ (v) in Proposition 4.3.5. □

Remark 4.5.8. (i) By [PS, pp. 5, 9], the $\mathscr{L}$-theory of real closed rings has a recursive axiomatization; in particular, both of the theories $T_{n,1}$ and $T_{n,2}$ also have recursive axiomatizations.

(ii) By Lemma 4.2.9 (III) (ii), every minimal prime ideal is parametrically definable in models of $T_{n,1}$ and in models of $T_{n,2}$, but minimal prime ideals cannot be defined without parameters in models of either of these theories: for instance, if V is a non-trivial real closed valuation ring with residue field $\boldsymbol{k}$, then $V \times_{\boldsymbol{k}} V \models T_{2,1}$ and the map $V \times_{\boldsymbol{k}} V \longrightarrow V \times_{\boldsymbol{k}} V$ given by $(a_1, a_2) \mapsto (a_2, a_1)$ is an automorphism which swaps the two minimal prime ideals.

(iii) If $A \models T_{n,1}$, then $\mathfrak{b}_A = \mathfrak{m}_A$ and thus the branching ideal $\mathfrak{b}_A$ is definable without parameters. If $A \models T_{n,2}$, then $\mathfrak{b}_A \subsetneq \mathfrak{m}_A$ and the branching ideal $\mathfrak{b}_A$ is also definable without parameters by the formula expressing the following about an element $a \in A$: "$a \in \mathrm{Ann}(a_i) + \mathrm{Ann}(a_j)$ for all non-zero pairwise orthogonal $a_1, \dots, a_n \in A$ and for all $i, j \in [n]$ such that $i \neq j$" (see Lemma 4.2.9 (III) (ii) and Lemma 4.4.7 (a)).

4.5.2 Model completeness

The core of this section consists of Theorems 4.5.15 and 4.5.21. The first key idea for these model completeness results is that if $A \subseteq B$ is an embedding (resp. a good embedding, see Definition 4.4.16) of rings of type $(n, 1)$ (resp. of rings of type $(n, 2)$), then the corresponding embeddings $A_i \subseteq B_i$ of non-trivial real closed valuation rings (see Convention 4.4.13) are elementary in suitable expansions of the language of rings (Proposition 2.3.13); the second key idea is using the first key idea together with Lemma 4.5.11 to show that under some hypotheses, A is existentially closed in B (Lemma 4.5.13 and Lemma 4.5.20), and then put everything together using Lemma 4.4.21 and Lemma 4.4.22.

Notation 4.5.9. Let $\{A_i\}_{i \in I}$ be a non-empty family of rings such that there exists a ring B and surjective ring homomorphisms $f_i : A_i \twoheadrightarrow B$ for all $i \in I$, and set $A := \prod_{B, i \in I} A_i$ (see Notation 4.4.6); in particular, A is a subdirect product of $\prod_{i \in I} A_i$ with canonical projection maps $p_i : A \twoheadrightarrow A_i$ for all $i \in I$.

(i) If $a \in A$, then write $a_i := p_i(a)$, and if $r \in \mathbb{N}^{\geq 2}$ and $\overline{a} \in A^r$, then write $\overline{a}_i := (a_{1i}, \dots, a_{ri}) \in A_i^r$ for all $i \in I$.

(ii) If $F(\overline{x}) \in A[\overline{x}]$, then write $F_i(\overline{x})$ for the polynomial in $A_i[\overline{x}]$ obtained by replacing each coefficient $a \in A$ appearing in $F(\overline{x})$ by a_i.

Remark 4.5.10. Let $\{A_i\}_{i \in I}$ and B be as in Notation 4.5.9 and set $A := \prod_{B, i \in I} A_i$. If $F(\overline{x}) \in A[x_1, \dots, x_r]$ and $\overline{a} \in A^r$, then $F(\overline{a})_i = F_i(\overline{a}_i)$ for all $i \in I$, therefore $A \models F(\overline{a}) = 0$ if and only if $A_i \models F_i(\overline{a}_i) = 0$ for all $i \in I$.

Lemma 4.5.11. *Let $\{A_i\}_{i \in I}$ and B be as in* Notation 4.5.9 *and set $A := \prod_{B, i \in I} A_i$. Let $\varphi(x_1, \dots, x_r)$ be a quantifier-free $\mathscr{L}$-formula with parameters from $S \subseteq A$, and*

$\overline{a} \in A^r$ be such that $A \models \varphi(\overline{a})$. There exist quantifier-free $\mathscr{L}$-formulas $\varphi_{\overline{a},i}(x_1, \ldots, x_r)$ with parameters from $S_i := p_i(S) \subseteq A_i$ $(i \in I)$ such that

(i) *$A \models \varphi(\overline{a})$ if and only if $A_i \models \varphi_{\overline{a},i}(\overline{a}_i)$ for all $i \in I$, and*

(ii) *if $\overline{a'} \in A^r$ is such that $A_i \models \varphi_{\overline{a},i}(\overline{a'}_i)$ for all $i \in I$, then $A \models \varphi(\overline{a'})$.*

Proof. Since $\varphi(x_1, \ldots, x_r)$ is quantifier-free and $A \models \varphi(\overline{a})$, it can be assumed that φ is of the form

$$\bigwedge\!\!\!\bigwedge_{\lambda \in \Lambda} F_\lambda^+(\overline{x}) = 0 \wedge\!\!\!\wedge F_\lambda^-(\overline{x}) \neq 0,$$

where Λ is a finite index set and $F_\lambda^\pm \in S[x_1, \ldots, x_r]$ for all $\lambda \in \Lambda$. For each $i \in I$, define $\varphi_{\overline{a},i}(x_1, \ldots, x_r)$ to be the $\mathscr{L}$-formula (with parameters from S_i)

$$\bigwedge\!\!\!\bigwedge_{\lambda \in \Lambda} F_{\lambda i}^+(\overline{x}) = 0 \wedge\!\!\!\wedge \bigwedge\!\!\!\bigwedge_{\lambda \in \Lambda} \{F_{\lambda i}^-(\overline{x}) \neq 0 \mid A_i \models F_{\lambda i}^-(\overline{a}_i) \neq 0\};$$

note that for each $i \in I$, if $A_i \models F_{\lambda i}^-(\overline{a}_i) = 0$ for all $\lambda \in \Lambda$, then $\varphi_{\overline{a},i}(\overline{x})$ is logically equivalent to $\bigwedge\!\!\!\bigwedge_{\lambda \in \Lambda} F_{\lambda i}^+(\overline{x}) = 0$. Items (i) and (ii) in the statement of the lemma now follow by Remark 4.5.10 and by construction of the formulas $\varphi_{\overline{a},i}(x_1, \ldots, x_r)$. □

Remark 4.5.12. The converse of item (ii) in Lemma 4.5.11 does not hold in general. For example, let $A := A_1 \times_{\boldsymbol{k}} A_2$, where A_1 and A_2 are non-trivial local domains with residue field $\boldsymbol{k}$, $\varphi(x)$ be the $\mathscr{L}$-formula $cx \neq 0$ with parameter c, where $c := (c_1, c_2) \in A \subseteq A_1 \times A_2$ is such that $c_i \in \mathfrak{m}_{A_i} \setminus \{0\}$, and define $a := (c_1, 0) \in A$; clearly $A \models \varphi(a)$. By the construction in the proof of Lemma 4.5.11, $\varphi_{a,1}(x)$ is the $\mathscr{L}$-formula $c_1 x \neq 0$ with parameter c_1, $\varphi_{a,2}(x)$ is an empty conjunct (hence logically equivalent to $x = x$), and clearly $A \models \varphi(a)$ if and only if $A_1 \models \varphi_{a,1}(a_1)$ and $A_2 \models \varphi_{a,1}(a_2)$; but $A \models \varphi(b)$ and $A_1 \not\models \varphi_{a,1}(b_1)$ for $b := (0, c_2) \in A$.

Model completeness for $T_{n,1}$

For the next lemma recall that any embedding $A \subseteq B$ of models of $T_{n,1}$ is local and it induces local embeddings $A_i \subseteq B_i$ for all $i \in [n]$, see Convention 4.4.13 and Lemma 4.4.14; moreover, if $A \subseteq B$ is sharp at $\mathfrak{b}_B$ $(= \mathfrak{m}_B)$, then $(A/\mathfrak{m}_A \cong)$ $A_i/\mathfrak{m}_{A_i} \cong B_i/\mathfrak{m}_{B_i}$ $(\cong B/\mathfrak{m}_B)$ for all $i \in [n]$, see Definition 4.4.18.

Lemma 4.5.13. *Let $A, B \models T_{n,1}$ be such that $A \subseteq B$.*

(I) *If* $A \subseteq B$ *is sharp at* $\mathfrak{b}_B$ $(= \mathfrak{m}_B)$, *then* A *is existentially closed in* B.

(II) *If both* A *and* B *are homogeneous* (Definition 4.4.19), *then* A *is existentially closed in* B.

Proof. Let $\varphi(x_1, \ldots, x_r)$ be a quantifier-free $\mathscr{L}$-formula with parameters from A and let $\bar{b} \in B^r$ be such that $B \models \varphi(\bar{b})$; by Lemma 4.5.11 there exist quantifier-free $\mathscr{L}$-formulas $\varphi_{\bar{b},i}(x_1, \ldots, x_r)$ with parameters from A_i $(i \in [n])$ such that

(i) $B \models \varphi(\bar{b})$ if and only if $B_i \models \varphi_{\bar{b},i}(\bar{b}_i)$ for all $i \in [n]$, and

(ii) if $\bar{a} \in B^r$ is such that $B_i \models \varphi_{\bar{b},i}(\bar{a}_i)$ for all $i \in [n]$, then $B \models \varphi(\bar{a})$.

(I). For each $i \in [n]$ and each $j \in [r]$, pick $c_{ji} \in A_i$ such that $c_{ji}/\mathfrak{m}_{A_i}$ is the image of $b_{ji}/\mathfrak{m}_{B_i}$ under the isomorphism $A_i/\mathfrak{m}_{A_i} \cong B_i/\mathfrak{m}_{B_i}$; since $A, B \models T_{n,1}$ and $\bar{b} \in B^r$, it follows by choice of $c_{ji} \in A_i$ that $c_j := (c_{j1}, \ldots, c_{jn}) \in A$ for all $j \in [r]$, and thus $\bar{c} := (c_1, \ldots, c_r) \in A^r$. Again by choice of $c_{ji} \in A_i$ and by item (i) above,

$$(B_i, \mathfrak{m}_{B_i}) \models \varphi_{\bar{b},i}(\bar{b}_i) \wedge\!\!\!\wedge \bigwedge_{j \in [r]} \mathtt{m}(b_{ji} - c_{ji});$$

since $A_i \subseteq B_i$ is a local embedding for all $i \in [n]$, by Proposition 2.3.13 (i) there exist $a_{ji} \in A_i$ $(j \in [r])$ such that

$$(A_i, \mathfrak{m}_{A_i}) \models \varphi_{\bar{b},i}(\bar{a}_i) \wedge\!\!\!\wedge \bigwedge_{j \in [r]} \mathtt{m}(a_{ji} - c_{ji})$$

for all $i \in [n]$, where $\bar{a}_i := (a_{1i}, \ldots a_{ri})$. Once again by choice of $c_{ji} \in A_i$, $a_j := (a_{j1}, \ldots, a_{jn}) \in A$ for all $j \in [r]$, and thus $\bar{a} := (a_1, \ldots, a_r) \in A^r \subseteq B^r$; since $A_i \models \varphi_{\bar{b},i}(\bar{a}_i)$ and $\varphi_{\bar{b},i}(\bar{x})$ is quantifier-free, $B_i \models \varphi_{\bar{b},i}(\bar{a_i})$ for $i \in [n]$, so by item (ii) above it follows that $B \models \varphi(\bar{a})$, and since A is a substructure of B and $\varphi(\bar{x})$ is quantifier-free, $A \models \varphi(\bar{a})$ follows.

(II). Since A and B are homogeneous, it can be assumed that there exist non-trivial real closed valuation rings V and W such that $A_i = V$ and $B_i = W$ for all $i \in [n]$; in particular, $b_{ji} \in W$ for all $i \in [n]$ and $j \in [r]$. By item (i) above,

$$(W, \mathfrak{m}_W) \models \bigwedge_{i \in [n]} \varphi_{\bar{b},i}(b_{1i}, \ldots, b_{ri}) \wedge\!\!\!\wedge \bigwedge_{j \in [r],\ i,i' \in [n]} \mathtt{m}(b_{ji} - b_{ji'}),$$

and since $V \subseteq W$ is a local embedding, by Proposition 2.3.13 (i) there exist $a_{ji} \in V$ ($i \in [n]$, $j \in [r]$) such that

$$(V, \mathfrak{m}_V) \models \bigwedge_{i\in[n]} \varphi_{\overline{b},i}(a_{1i}, \dots, a_{ri}) \wedge \bigwedge_{j\in[r],\ i,i'\in[n]} \mathtt{m}(a_{ji} - a_{ji'}).$$

It follows that for each $j \in [r]$, $a_j := (a_{j1}, \dots, a_{jn}) \in A$ and thus $\overline{a} := (a_1, \dots, a_r) \in A^r \subseteq B^r$; since $V \models \varphi_{\overline{b},i}(\overline{a}_i)$ and $\varphi_{\overline{b},i}(\overline{x})$ is quantifier-free, $W \models \varphi_{\overline{b},i}(\overline{a}_i)$ for $i \in [n]$, so by item (ii) above it follows that $B \models \varphi(\overline{a})$, and since A is a substructure of B and $\varphi(\overline{x})$ is quantifier-free, $A \models \varphi(\overline{a})$ follows. □

Remark 4.5.14. The proof of Lemma 4.5.13 (I) can be used *mutatis mutandis* to show that given arbitary collections of non-trivial real closed valuation rings $\{V_i\}_{i\in I}$ and $\{W_i\}_{i\in I}$ with $V_i/\mathfrak{m}_{V_i} \cong W_i/\mathfrak{m}_{W_i} =: \boldsymbol{k}$ and $V_i \subseteq W_i$ for all $i \in I$, then $\prod_{\boldsymbol{k},i\in I} V_i$ is existentially closed in $\prod_{\boldsymbol{k},i\in I} W_i$.

Theorem 4.5.15. *$T_{n,1}$ is model complete.*

Proof. Combine Lemmas 4.5.13 and 4.4.21. □

Model completeness for $T_{n,2}$

The main difference between embeddings of models of $T_{n,1}$ and embeddings of models of $T_{n,2}$ is that every embedding $A \subseteq B$ of models of $T_{n,1}$ is local (hence also $\mathfrak{b}_B \cap A = \mathfrak{b}_A$), but this is not the case for models of $T_{n,2}$ (i.e., not every embedding of models of $T_{n,2}$ is a good embedding, Example 4.4.15 and Definition 4.4.16); this fact, together with the next lemma implies that $T_{n,2}$ is not model complete in the language of rings:

Lemma 4.5.16. *Let A and B be local real closed rings of rank n such that $A \subseteq B$. If A is existentially closed in B, then the embedding $A \subseteq B$ is local and $\mathfrak{q} \cap A$ is a branching ideal of A for every branching ideal $\mathfrak{q} \in \mathrm{Spec}(B)$.*

Proof. That $A \subseteq B$ must be a local embedding is clear. Let now $a_1, \dots, a_n \in A$ be non-zero pairwise orthogonal elements, so that $\mathrm{Spec}^{\min}(A) = \{\mathrm{Ann}_A(a_i) \mid i \in [n]\}$ and $\mathrm{Spec}^{\min}(B) = \{\mathrm{Ann}_B(a_i) \mid i \in [n]\}$ (Corollary 4.2.11). Pick a branching ideal $\mathfrak{q} \in \mathrm{Spec}(B)$; by Remark 4.3.6 there exist $i, j \in [n]$ with $i \neq j$ such that $\mathfrak{q} = \mathrm{Ann}_B(a_i) + \mathrm{Ann}_B(a_j)$, therefore $\mathrm{Ann}_A(a_i) + \mathrm{Ann}_A(a_j) \subseteq \mathfrak{q} \cap A$. Pick now $b \in \mathfrak{q} \cap A$; then

$$B \models \exists xy[xa_i = 0 \wedge ya_j = 0 \wedge b = x + y],$$

and since $a_i, a_j, b \in A$ and A is existentially closed in B, it follows that

$$A \models \exists xy[xa_i = 0 \wedge ya_j = 0 \wedge b = x + y],$$

therefore $b \in \mathrm{Ann}_A(a_i) + \mathrm{Ann}_A(a_j)$ and thus $\mathfrak{q} \cap A = \mathrm{Ann}_A(a_i) + \mathrm{Ann}_A(a_j) \in \mathrm{Spec}(A)$ is a branching ideal. □

Example 4.4.15 shows that there exist embeddings of models of $T_{n,2}$ which are not local, and also that there exist embeddings of models of $T_{n,2}$ which do not map the branching ideal to the branching ideal; in view of Lemma 4.5.16, to obtain model completeness for $T_{n,2}$ one must enlarge the language of rings is such a way that every embedding of models of $T_{n,2}$ in the resulting language is a good embedding.

Definition 4.5.17. (i) Let $\mathtt{b}$ and $\mathtt{m}$ be two unary predicates and define $\mathscr{L}^* := \mathscr{L}(\mathtt{b}, \mathtt{m})$.

(ii) Let $T^*_{n,2}$ be the $\mathscr{L}^*$-theory $T_{n,2}$ together with the sentence expressing that $\mathtt{b}$ and $\mathtt{m}$ are interpreted as the branching ideal (Remark 4.5.8 (iii)) and as the set of non-units, respectively.

Remark 4.5.18. Let $A := \prod_{C, i \in [n]} A_i \models T^*_{n,2}$, so that $A_1, \ldots, A_n$ and C are non-trivial real closed valuation rings such that for each $i \in [n]$ there exists a surjective ring homomorphism $A_i \longrightarrow\!\!\!\!\!\rightarrow C$ onto a non-trivial real closed valuation ring C.

(i) For each $i \in [n]$, A_i is regarded as an $\mathscr{L}^*$-structure in the canonical way, that is, $\mathtt{b}(A_i) = \mathfrak{b}_{A_i} := \ker(A_i \twoheadrightarrow C)$ and $\mathtt{m}(A_i) := \mathfrak{m}_{A_i}$; in particular, the projection map $A \longrightarrow\!\!\!\!\!\rightarrow A_i$ is an $\mathscr{L}^*$-homomorphism.

(ii) Note that $\mathfrak{b}_A = \mathfrak{b}_{A_1} \times \cdots \times \mathfrak{b}_{A_n}$ and $\mathfrak{m}_A = \mathfrak{m}_{A_1} \times \cdots \times \mathfrak{m}_{A_n}$ when $\mathfrak{b}_A$ and $\mathfrak{m}_A$ are regarded as subsets of $\prod_{i=1}^n A_i$; in particular, if $F(\overline{x}) \in A[x_1, \ldots, x_r]$ and $\overline{a} \in A^r$, then $A \models \mathtt{b}(F(\overline{a}))$ if and only if $A_i \models \mathtt{b}(F_i(\overline{a}_i))$ for all $i \in [n]$, and $A \models \mathtt{m}(F(\overline{a}))$ if and only if $A_i \models \mathtt{m}(F_i(\overline{a}_i))$ for all $i \in [n]$ (cf. Remark 4.5.10).

Lemma 4.5.19. *Let $A := \prod_{C, i \in [n]} A_i \models T^*_{n,2}$ and $\varphi(x_1, \ldots, x_r)$ be a quantifier-free $\mathscr{L}^*$-formula with parameters from $S \subseteq A$, and $\overline{a} \in A^r$ be such that $A \models \varphi(\overline{a})$. There exist quantifier-free $\mathscr{L}^*$-formulas $\varphi_{\overline{a},i}(x_1, \ldots, x_r)$ with parameters from $S_i := p_i(S) \subseteq A_i$ $(i \in [n])$ such that*

(i) $A \models \varphi(\overline{a})$ *if and only if* $A_i \models \varphi_{\overline{a},i}(\overline{a}_i)$ *for all* $i \in [n]$, *and*

(ii) *if* $\overline{b} \in A^r$ *is such that* $A_i \models \varphi_{\overline{a},i}(\overline{b}_i)$ *for all* $i \in [n]$, *then* $A \models \varphi(\overline{b})$.

Proof. Analogous to the proof of Lemma 4.5.11 using Remark 4.5.18 (ii). □

Lemma 4.5.20. *Let* $A, B \models T^*_{n,2}$ *be such that* $A \subseteq B$ *as* $\mathscr{L}^*$*-structures.*

(I) *If* $A \subseteq B$ *is sharp at* $\mathfrak{b}_B$, *then* A *is existentially closed in* B *as an* $\mathscr{L}^*$*-structure.*

(II) *If both* A *and* B *are homogeneous, then* A *is existentially closed in* B *as an* $\mathscr{L}^*$*-structure.*

Proof. Analogous to the proof of Lemma 4.5.13 using Lemma 4.5.19 and Proposition 2.3.13 (ii). □

Theorem 4.5.21. $T^*_{n,2}$ *is model complete.*

Proof. Combine Lemmas 4.5.20 and 4.4.22. □

4.5.3 Consequences of model completeness

The model companion of local real closed (SV-) rings of rank n

For the next result, recall that the class of local real closed rings of rank n is elementary in the language of rings $\mathscr{L}$; explicitly, an axiomatization for this class of rings is given by the axioms for local real closed rings ([PS]) together with the $\mathscr{L}$-sentence $\varphi_{\mathrm{rk}=n}$ defined in Lemma 4.5.1.

Corollary 4.5.22. $T_{n,1}$ *is the model companion of* T_n *and also of the* $\mathscr{L}$*-theory of local real closed rings of rank* n.

Proof. Combine Theorem 4.5.15, Lemma 4.4.10, and Proposition 4.4.12. □

Example 4.5.23. T_n does not have the amalgamation property; in particular, $T_{n,1}$ is not the model completion of T_n ([CK90, Proposition 3.5.18]). Indeed, let V be a non-trivial real closed valuation ring of Krull dimension at least 2 and with residue field $\boldsymbol{k}$, and let $\mathfrak{p}$ be a non-zero non-maximal prime ideal of V. Define $A := \prod^n_{V/\mathfrak{p}} V$, $B := \prod^n_{V_\mathfrak{p}/\mathfrak{p}V_\mathfrak{p}} V_\mathfrak{p}$, and $C := \prod^n_{\boldsymbol{k}} V$; note in particular that $A \models T_{n,2}$, so that the branching ideal $\mathfrak{b}_A$ of A is property contained in $\mathfrak{m}_A$, and also $B, C \models T_{n,1}$. Clearly

$A \subseteq B, C$, and if T_n has the amalgamation property, then by Corollary 4.5.22 there exists $D \models T_{n,1}$ amalgamating B and C over A; by Lemma 4.4.14, $B \cap \mathfrak{m}_D = \mathfrak{m}_B$ and $C \cap \mathfrak{m}_D = \mathfrak{m}_C$, but $A \cap \mathfrak{m}_B = \ker(A \twoheadrightarrow V/\mathfrak{p}) = \mathfrak{b}_A$ and $A \cap \mathfrak{m}_C = \ker(A \twoheadrightarrow \boldsymbol{k}) = \mathfrak{m}_A$, therefore $\mathfrak{b}_A = A \cap \mathfrak{m}_D = \mathfrak{m}_A$, a contradiction to $\mathfrak{b}_A \neq \mathfrak{m}_A$.

Lemma 4.5.24. *Let $A \models T_n$ and $B, C \models T_{n,1}$ be such that $A \subseteq B, C$. If $A \subseteq B$ and $A \subseteq C$ are local embeddings, then there exists $D \models T_{n,1}$ amalgamating B and C over A.*

Proof. Write $\operatorname{Spec}^{\min}(A) = \{\mathfrak{p}_{A,i} \mid i \in [n]\}$, $\operatorname{Spec}^{\min}(B) = \{\mathfrak{p}_{B,i} \mid i \in [n]\}$, and $\operatorname{Spec}^{\min}(C) = \{\mathfrak{p}_{C,i} \mid i \in [n]\}$, and assume without loss of generality that $\mathfrak{p}_{B,i} \cap A = \mathfrak{p}_{A,i} = \mathfrak{p}_{C,i} \cap A$ for all $i \in [n]$ (Corollary 4.2.11). Since $A \subseteq B$ and $A \subseteq C$ are local embeddings, $A/\mathfrak{p}_{A,i} \subseteq B/\mathfrak{p}_{B,i}$ and $A/\mathfrak{p}_{A,i} \subseteq C/\mathfrak{p}_{C,i}$ are local embeddings for all $i \in [n]$ (Remark 4.2.12), therefore by Lemma 2.3.14 there exist non-trivial real closed valuation rings V_i amalgamating $B/\mathfrak{p}_{B,i}$ and $C/\mathfrak{p}_{C,i}$ over $A/\mathfrak{p}_{A,i}$ as $\mathscr{L}(\mathtt{m})$-structures. Since $\mathsf{RCVR}(\mathtt{m})$ is complete (Proposition 2.3.13 (i)), there exists a non-trivial real closed valuation ring V with residue field $\boldsymbol{k}$ such that $(V_i, \mathfrak{m}_{V_i}) \subseteq (V, \mathfrak{m}_V)$ for all $i \in [n]$, therefore it follows that the canonical composite embeddings

$$A \subseteq B \hookrightarrow \prod_{i\in[n]} B/\mathfrak{p}_{B,i} \subseteq \prod_{i\in[n]} V_i \subseteq \prod_{i\in[n]} V \quad \text{and} \quad A \subseteq C \hookrightarrow \prod_{i\in[n]} C/\mathfrak{p}_{C,i} \subseteq \prod_{i\in[n]} V_i \subseteq \prod_{i\in[n]} V$$

corestrict to composite embeddings $A \subseteq B \hookrightarrow \prod_{\boldsymbol{k}}^n V$ and $A \subseteq C \hookrightarrow \prod_{\boldsymbol{k}}^n V$ (respectively) witnessing that $\prod_{\boldsymbol{k}}^n V \models T_{n,1}$ amalgamates B and C over A. □

Proposition 4.5.25. *Let $T_{n,1}(\mathtt{m})$ and $T_n(\mathtt{m})$ be the $\mathscr{L}(\mathtt{m})$-theories $T_{n,1}$ and T_n together with the sentence expressing that $\mathtt{m}$ is the set of non-units, respectively.*

(i) *The models of $T_n(\mathtt{m})$ have prime extensions in the class of models of $T_{n,1}(\mathtt{m})$.*

(ii) *$T_{n,1}(\mathtt{m})$ is the model completion of $T_n(\mathtt{m})$.*

Proof. Note first that since $T_{n,1}(\mathtt{m})$ is an extension by definitions of $T_{n,1}$ and this latter theory is model complete, $T_{n,1}(\mathtt{m})$ is also model complete.

(i). Let $A \models T_n(\mathtt{m})$ and set $\operatorname{Spec}^{\min}(A) = \{\mathfrak{p}_i \mid i \in [n]\}$. By Lemma 4.4.10, A embeds into $A' := \prod_{A/\mathfrak{m}_A, i\in[n]} A/\mathfrak{p}_i \models T_{n,1}(\mathtt{m})$ as an $\mathscr{L}(\mathtt{m})$-structure, and it is claimed that A' is the prime extension of A in the class of models of $T_{n,1}(\mathtt{m})$. Let $B \models T_{n,1}(\mathtt{m})$

and suppose that $A \subseteq B$ as $\mathscr{L}(\mathtt{m})$-structures; set $\mathrm{Spec}^{\min}(B) = \{\mathfrak{q}_i \mid i \in [n]\}$, so that the canonical embedding $B \longrightarrow \prod_{i\in[n]} B/\mathfrak{q}_i$ corestricts to an isomorphism $B \cong \prod_{B/\mathfrak{m}_B, i\in[n]} B/\mathfrak{q}_i$ (Lemma 4.4.7 (b)); since the embedding $A \subseteq B$ is local, it induces local embeddings $A/\mathfrak{p}_i \subseteq B/\mathfrak{q}_i$ for all $i \in [n]$, from which it follows that the induced embedding $\prod_{i\in[n]} A/\mathfrak{p}_i \longrightarrow \prod_{i\in[n]} B/\mathfrak{q}_i$ restricts to an $\mathscr{L}(\mathtt{m})$-embedding $A' \longrightarrow B$ over A.

(ii). By Corollary 4.5.22 and [CK90, Proposition 3.5.18] it suffices to show that if $B, C \models T_{n,1}(\mathtt{m})$ contain a common $\mathscr{L}(\mathtt{m})$-substructure $A \subseteq B, C$ such that $A \models T_n(\mathtt{m})$, then there exists $D \models T_{n,1}(\mathtt{m})$ amalgamating B and C over A; the existence of such D follows by Lemma 4.5.24. □

Remark 4.5.26. Let T'_n be the $\mathscr{L}$-theory T_n together with the sentence expressing that every non-unit is a sum of two zero divisors. By the equivalence (i) ⇔ (v) in Proposition 4.3.5, models of T'_n are local real closed SV-rings of rank n whose maximal ideal is a branching ideal; clearly any model of $T_{n,1}$ is a model of T'_n, therefore $T_{n,1}$ is also the model companion of T'_n by Corollary 4.5.22. But in this case, $T_{n,1}$ is even the model completion of T'_n in the language of rings: indeed, if $B, C \models T_{n,1}$ contain a common $\mathscr{L}$-substructure $A \subseteq B, C$ such that $A \models T'_n$, then by Lemma 4.3.8 both $A \subseteq B$ and $A \subseteq C$ are local embeddings, therefore by Lemma 4.5.24 there exists $D \models T_{n,1}$ amalgamating B and C over A.

Completeness, decidability, and NIP

Corollary 4.5.27. *The theories $T_{n,1}$ and $T_{n,2}$ are complete.*

Proof. By Theorems 4.5.15 and 4.5.21 to prove completeness it suffices to show that each of the theories $T_{n,1}$ and $T^*_{n,2}$ have the joint embedding property; note that $T^*_{n,2}$ is an extension by definitions of $T_{n,2}$, so completeness of $T^*_{n,2}$ entails completeness of $T_{n,2}$. Let $A, B \models T_{n,1}$; by Lemma 4.4.21, it can be assumed that both A and B are homogeneous, so that there exist non-trivial real closed valuation rings V and V' with residue fields $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively, such that $A = \prod^n_{\boldsymbol{k}} V$ and $B = \prod^n_{\boldsymbol{k}'} V'$. Since $\mathsf{RCVR}(\mathtt{m})$ is complete (Proposition 2.3.13 (i)), there exists a non-trivial real closed valuation ring W with residue field $\boldsymbol{l}$ and local embeddings $V, V' \subseteq W$, from which it follows that both A and B embed into $\prod^n_{\boldsymbol{l}} W \models T_{n,1}$; the joint embedding property for $T^*_{n,2}$ follows in a similar manner appealing to Lemma 4.4.22 and Proposition 2.3.13 (ii). □

Corollary 4.5.28. *The theories $T_{n,1}$ and $T_{n,2}$ are decidable.*

Proof. Since $T_{n,1}$ and $T_{n,2}$ are complete by Corollary 4.5.27 and recursively axiomatizable (Remark 4.5.8 (i)), they are decidable. □

For the last result of this subsection, recall that if A is an NIP $\mathscr{L}_1$-structure ([Sim15]) and B is an $\mathscr{L}_2$-structure which is interpretable in A ([Hod93, Section 5.3]), then B is also NIP.

Corollary 4.5.29. *The theories $T_{n,1}$ and $T_{n,2}$ are NIP.*

Proof. Since IP is preserved under elementary equivalence, it suffices to show by Corollary 4.5.27 that $\prod^n_{\boldsymbol{k}} V \models T_{n,1}$ and $\prod^n_{V/\mathfrak{p}} V \models T_{n,2}$ are NIP, where V is a non-trivial real closed valuation ring of Krull dimension at least 2 with residue field $\boldsymbol{k}$ and $\mathfrak{p}$ is a non-zero non-maximal prime ideal of V. Since weakly o-minimal theories are NIP ([Sim15, Appendix A.1.3]), $\mathsf{RCVR}(\leq, \mathrm{div})$ is NIP by [Dic87, Corollary 1.6 (c)], therefore so is $V \models \mathsf{RCVR}$. By [Sim15, Proposition 3.23], the Shelah expansion V^{Sh} is also NIP, and since prime ideals of rings are externally definable ([dHJ24, Fact 4.1]), it follows that both structures $(V, \mathfrak{m}_V)$ and $(V, \mathfrak{p})$ are NIP; since $\prod^n_{\boldsymbol{k}} V$ is interpretable in $(V, \mathfrak{m}_V)$ and $\prod^n_{V/\mathfrak{p}} V$ is interpretable in $(V, \mathfrak{p})$, it follows that both $\prod^n_{\boldsymbol{k}} V$ and $\prod^n_{V/\mathfrak{p}} V$ are NIP, as required. □

Since all local real closed SV-rings of rank 2 have exactly one branching ideal, it follows from the above that the $\mathscr{L}$-theory of local real closed SV-rings of rank 2 splits into two complete, decidable, and NIP $\mathscr{L}$-theories, namely, the theories $T_{n,1}$ and $T_{n,2}$; the results above together with this observation build up to the following:

Conjecture 4.5.30. *The $\mathscr{L}$-theory T_n of local real closed SV-rings of rank $n \in \mathbb{N}^{\geq 2}$ splits into finitely many complete, decidable, and NIP $\mathscr{L}$-theories.*

A proof of the above conjecture would rest in a good model-theoretic understanding of arbitrary local real closed SV-rings of finite rank. Subsection 4.6.1 highlights some of the key obstacles that arise in the model-theoretic analysis of such rings when there are two or more branching ideals; based on the results in this section, Conjecture 4.6.9

gives candidates for what each of the finitely many completions of T_n mentioned in Conjecture 4.5.30 could be.

4.5.4 Quantifier elimination for $T_{n,1}$

For this last section recall that any real closed ring A is an f-ring $(A, \vee, \wedge, \leq)$, and since $a \geq 0$ in A if and only if a is a square, the partial order $\leq$ on A is definable, and thus so are the lattice operations $\vee$ and $\wedge$. In particular, any $\mathscr{L}$-theory T of real closed rings can be regarded as an $\mathscr{L}(\vee, \wedge, \leq)$-theory via extension by definitions; moreover, if $A \subseteq B$ is an $\mathscr{L}$-embedding of real closed rings, then $A \subseteq B$ is also an $\mathscr{L}(\vee, \wedge, \leq)$-embedding (the proof of this fact is contained in [PS, p. 11]). The next example, which is inspired by the example in [PS, p. 19], shows failure of quantifier elimination for the theory $T_{n,1}$ in various languages.

Example 4.5.31. If $\mathscr{L}'$ is any language such that $\mathscr{L} \subseteq \mathscr{L}' \subseteq \mathscr{L}(\vee, \wedge, \leq, \mathrm{div})$, then $T_{n,1}$ does not have quantifier elimination in $\mathscr{L}'$; for notational simplicity only the case $n = 2$ will be considered here, but the construction can be easily adapted for arbitrary $n \in \mathbb{N}^{\geq 2}$. Assume for contradiction that $T_{2,1}$ has quantifier elimination in $\mathscr{L}'$ and let V be a non-trivial real closed valuation ring of Krull dimension at least 2 with residue field $\boldsymbol{k}$. Let $\mathfrak{p}$ be a non-zero non-maximal prime ideal of V, and define $B := V \times_{\boldsymbol{k}} V$ and $C := V \times_{\boldsymbol{k}} V/\mathfrak{p}$; then the maps $\varepsilon_B : V \longrightarrow B$ and $\varepsilon_C : V \longrightarrow C$ given by $\varepsilon_B(v) := (v, v)$ and $\varepsilon_C(v, v/\mathfrak{p})$ are $\mathscr{L}'$-embeddings, and since $T_{2,1}$ is model complete (Theorem 4.5.15) there exist $D \models T_{2,1}$ and $\mathscr{L}'$-embeddings $f_B : B \hookrightarrow D$ and $f_C : C \hookrightarrow D$ such that the diagram

$$\begin{array}{ccc} B & \overset{f_B}{\hookrightarrow} & D \\ {\scriptstyle \varepsilon_B}\big\uparrow & & \big\uparrow{\scriptstyle f_C} \\ V & \underset{\varepsilon_C}{\hookrightarrow} & C \end{array}$$

commutes ([CK90, Proposition 3.5.19]). Pick $\eta \in \mathfrak{m}_V \setminus \mathfrak{p}$; then $b := (0, \eta) \in B$ and $b' := (\eta, 0) \in B$ are non-zero orthogonal elements, and $c := (0, \eta/\mathfrak{p}) \in C$ and $c' := (\eta, 0/\mathfrak{p}) \in C$ are non-zero orthogonal elements, therefore $\mathrm{Spec}^{\min}(B) = \{\mathrm{Ann}_B(b), \mathrm{Ann}_B(b')\}$, $\mathrm{Spec}^{\min}(C) = \{\mathrm{Ann}_C(c), \mathrm{Ann}_C(c')\}$ and

$$\mathrm{Spec}^{\min}(D) = \{\mathrm{Ann}_D(b), \mathrm{Ann}_D(b')\} = \{\mathrm{Ann}_D(c), \mathrm{Ann}_D(c')\}$$

by Corollary 4.2.11. But then

$$\mathrm{Spec}(f_B \circ \varepsilon_B)(\mathrm{Ann}_D(b)) = \mathrm{Spec}(f_B \circ \varepsilon_B)(\mathrm{Ann}_D(b')) = \mathrm{Spec}(f_C \circ \varepsilon_C)(\mathrm{Ann}_D(c')) = (0)$$

and $\mathrm{Spec}(f_C \circ \varepsilon_C)(\mathrm{Ann}_D(c)) = \mathfrak{p}$, therefore the square of Zariski spectra induced by the commutative square above does not commute, giving the required contradiction.

The example above shows that $T_{n,1}$ fails to have quantifier elimination in any language $\mathscr{L}'$ such that $\mathscr{L} \subseteq \mathscr{L}' \subseteq \mathscr{L}(\vee, \wedge, \leq, \mathrm{div})$ due to the fact that any $\mathscr{L}'$-structure A such that $A \models T_{n,1}$ has $\mathscr{L}'$-substructures of smaller rank; enlarging the language to force all substructures to have rank n turns out to be sufficient to obtain quantifier elimination.

Definition 4.5.32. Let $e_1, \dots, e_n$ be new constant symbols and let $\mathscr{L}^\dagger := \mathscr{L}(\vee, \wedge, \leq, \mathrm{div}, e_1, \dots, e_n)$. Define $T^\dagger_{n,1}$ to be the canonical extension by definitions of $T_{n,1}$ to $\mathscr{L}(\vee, \wedge, \leq, \mathrm{div})$ together with the statement expressing that $e_1, \dots, e_n$ are non-zero and pairwise orthogonal elements.

Theorem 4.5.33. *$T^\dagger_{n,1}$ has quantifier elimination.*

Proof. Let $A, B, C \models (T^\dagger_{n,1})_\forall$ be such that $A \subseteq B, C$ as $\mathscr{L}^\dagger$-structures. Since $T_{n,1}$ is model complete by Theorem 4.5.15, so is $T^\dagger_{n,1}$, therefore it suffices to show that there exists $D \models T^\dagger_{n,1}$ amalgamating B and C over A as $\mathscr{L}^\dagger$-structures; moreover, since $B, C \models (T^\dagger_{n,1})_\forall$, it can be assumed that $B, C \models T^\dagger_{n,1}$.

Claim 1. A is a reduced local SV-f-ring of rank n (see [Sch10b, Definition 4.1]).

Proof of Claim 1. A is a reduced f-ring by the equivalence (i) $\Leftrightarrow$ (ii) in [BKW77, Theoreme 9.1.2], and A is local because B is local and this property can be expressed by the universal $\mathscr{L}^\dagger$-sentence $\forall x[\mathrm{div}(x, 1) \text{ or } \mathrm{div}(1 - x, 1)]$. Since A is reduced and local, its rank is the supremum of the lengths of sequences of non-zero pairwise orthogonal elements (Lemma 4.2.9 (III) (i)), and as A is an $\mathscr{L}^\dagger$-substructure of $B \models T^\dagger_{n,1}$, it follows that $\mathrm{rk}(A) = n$; finally, A is an SV-ring by Lemma 4.5.2, and A has bounded inversion because B has bounded inversion ([SM99, Proposition 12.4]) and this property can be expressed by the universal $\mathscr{L}^\dagger$-sentence $\forall x[1 \leq x \to \mathrm{div}(x, 1)]$, therefore A is an SV-f-ring by [DM95, Proposition 3.2] and [Sch10a, Proposition 3.3]. $\square_{\text{Claim 1}}$

Set $\mathfrak{p}_{A,i} := \mathrm{Ann}_A(a_i)$, $\mathfrak{p}_{B,i} := \mathrm{Ann}_B(a_i)$, and $\mathfrak{p}_{C,i} := \mathrm{Ann}_C(a_i)$ for all $i \in [n]$, where each $a_i \in A$ is the interpretation of the constant symbol $e_i \in \mathscr{L}^\dagger$, so that $\mathrm{Spec}^{\min}(A) =$

$\{\mathfrak{p}_{A,i} \mid i \in [n]\}$, $\mathrm{Spec}^{\min}(B) = \{\mathfrak{p}_{B,i} \mid i \in [n]\}$, and $\mathrm{Spec}^{\min}(C) = \{\mathfrak{p}_{C,i} \mid i \in [n]\}$ by Corollary 4.2.11. Since $\mathfrak{p}_{A,i}$ is a minimal prime ideal of the reduced f-ring A, $\mathfrak{p}_{A,i}$ an irreducible ℓ-ideal ([BKW77, Sections 8.4 and 8.5, Theoreme 9.3.2]), therefore $A/\mathfrak{p}_{A,i}$ is a totally ordered domain and the residue map $A \twoheadrightarrow A/\mathfrak{p}_{A,i}$ a homomorphism of lattice-ordered rings (see also [SM99, pp. 31–33]).

Claim 2. The ring embeddings $A/\mathfrak{p}_{A,i} \subseteq B/\mathfrak{p}_{B,i}, C/\mathfrak{p}_{C,i}$ are $\mathscr{L}(\leq, \mathtt{m})$-embeddings for all $i \in [n]$.

Proof of Claim 2. The $\mathscr{L}^\dagger$-embeddings $A \subseteq B, C$ are local embeddings (Remark 2.3.12), therefore they induce local embeddings $A/\mathfrak{p}_{A,i} \subseteq B/\mathfrak{p}_{B,i}, C/\mathfrak{p}_{C,i}$ for all $i \in [n]$ (Remark 4.2.12), and thus it remains to show that these latter embeddings are $\mathscr{L}(\leq)$-embeddings. Pick $a \in A$; then $a \vee 0 \in A \subseteq B$, $(a \vee 0)/\mathfrak{p}_{A,i} = (a/\mathfrak{p}_{A,i}) \vee (0/\mathfrak{p}_{A,i})$, and $(a \vee 0)/\mathfrak{p}_{B,i} = (a/\mathfrak{p}_{B,i}) \vee (0/\mathfrak{p}_{B,i})$, from which it follows that $a/\mathfrak{p}_{A,i} \geq 0/\mathfrak{p}_{A,i}$ if and only if $a/\mathfrak{p}_{B,i} \geq 0/\mathfrak{p}_{B,i}$, as required. $\square_{\text{Claim 2}}$

By Claim 1 and by the above, $A/\mathfrak{p}_{A,i}$ is a totally ordered valuation ring for all $i \in [n]$, and since $B, C \models T^\dagger_{n,1}$, $B/\mathfrak{p}_{B,i}$ and $C/\mathfrak{p}_{C,i}$ are non-trivial real closed valuation rings, therefore by Claim 2 and Lemma 2.3.14 there exist non-trivial real closed valuation rings V_i and local embeddings $B/\mathfrak{p}_{B,i}, C/\mathfrak{p}_{C,i} \subseteq V_i$ such that the diagram

$$\begin{array}{ccc} B/\mathfrak{p}_{B,i} & \hookrightarrow & V_i \\ \uparrow & & \uparrow \\ A/\mathfrak{p}_{A,i} & \hookrightarrow & C/\mathfrak{p}_{C,i} \end{array}$$

commutes for all $i \in [n]$. Amalgamate all V_i into a single non-trivial real closed valuation ring V with residue field $\boldsymbol{k}$ in such a way that all the embeddings $V_i \subseteq V$ are local and define $D_0 := \prod^n_{\boldsymbol{k}} V \models T_{n,1}$. Note that $a_i \in \mathfrak{p}_{A,j}$ if and only if $j = i$ for all $i, j \in [n]$ and the images of $a_i/\mathfrak{p}_{A,i}$ and $a_i/\mathfrak{p}_{C,i}$ in V coincide; it follows that the ring D_0 can be expanded to a model $D \models T^\dagger_{n,1}$ in such a way that the composite local embeddings $B/\mathfrak{p}_{B,i} \subseteq V_i \subseteq V$ and $C/\mathfrak{p}_{C,i} \subseteq V_i \subseteq V$ induce $\mathscr{L}^\dagger$-embeddings $B \subseteq D$ and $C \subseteq D$ (note that the ring embeddings $B, C \subseteq D$ are $\mathscr{L}(\mathrm{div})$-embeddings by

Theorem 4.5.15) making the diagram

$$\begin{array}{ccc} B & \hookrightarrow & D \\ \uparrow & & \uparrow \\ A & \hookrightarrow & C \end{array}$$

commute, concluding thus the proof. □

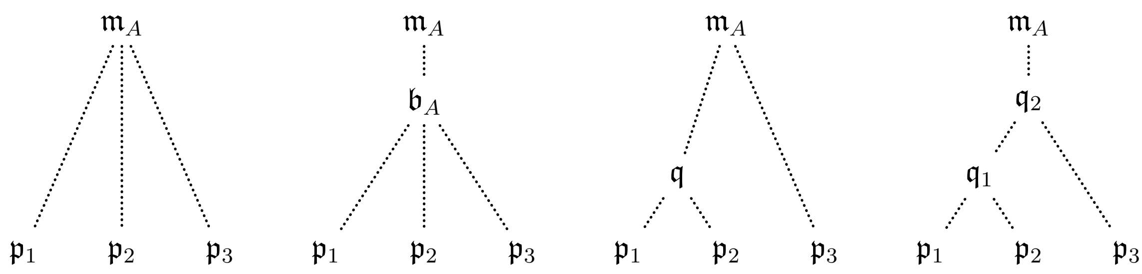
4.6 Beyond one branching ideal

The aim of this section is twofold. To start with, Subsection 4.6.1 contains the key difficulties that arise in trying to carry the same model-theoretic analysis as done in Section 4.5 to those local real closed SV-rings of finite rank which have two or more branching ideals; then, Subsection 4.6.2 introduces the notion of the branching spectrum of a local real closed ring of finite rank in order to connect the results of the previous section with the model theory of real closed rings with radical relations as developed in [PS] (see also [Gui25]) with a view towards overcoming the obstacles described Subsection 4.6.1 as well as laying down the path towards a possible proof of Conjecture 4.5.30.

As in Section 4.5, fix $n \in \mathbb{N}^{\geq 2}$ for what remains; all model-theoretic statements are again assumed to be phrased with respect to the language of rings $\mathscr{L}$ unless stated otherwise.

4.6.1 Difficulties with two or more branching ideals

A recurring theme in the study of real closed rings is that their Zariski spectrum serves as a rough measure of their complexity, and this has already been seen in the previous two sections with the differences between rings of type $(n, 1)$ and of type $(n, 2)$. Note that by Remarks 4.3.2 (ii) and 4.3.7, a local real closed ring of rank 2 has exactly one branching ideal, and if A is a local real closed ring of rank 3, then all the possible configurations of its minimal prime ideals, branching ideals, and its maximal ideal can be summarized in the following four Hasse diagrams, where the dotted lines indicate that there could be more prime ideals between the two nodes that each dotted line connects:

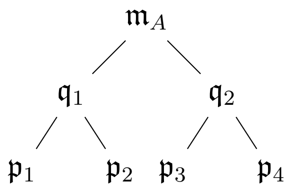


If A is furthermore an SV-ring, then the fist diagram indicates that A is a ring of type $(3, 1)$, and the second diagram indicates that A is a ring of type $(3, 2)$. The fact that in the two diagrams on the left there is exactly one branching ideal creates a situation of "symmetry" that has been heavily exploited in Section 4.5, namely, the model-theoretic analysis of rings of type (n, j) ($j \in [2]$) is done in terms of the model theory of each of the residue domains $A/\mathfrak{p}_1, \ldots, A/\mathfrak{p}_n$ ($\mathrm{Spec}^{\min}(A) = \{\mathfrak{p}_i \mid i \in [n]\}$), where each of these domains are all canonically regarded as structures in the same language ($\mathscr{L}(\mathtt{m})$ if $j = 1$ and $\mathscr{L}(\mathtt{b}, \mathtt{m})$ if $j = 2$, see Subsection 4.5.2) and can therefore be easily compared and manipulated as models of a suitable theory ($\mathsf{RCVR}(\mathtt{m})$ if $j = 1$ and $\mathsf{RCVR}(\mathtt{b}, \mathtt{m})$ if $j = 2$).

This "symmetry" can break in the presence of more than one branching ideal. In particular, two residue domains $A/\mathfrak{p}_i$ and $A/\mathfrak{p}_j$ may carry information about a different number of prime ideals of A, and thus capturing this information via unary predicates leads to different languages (e.g., in the case of the third diagram, $A/\mathfrak{p}_2$ can be canonically regarded as an $\mathscr{L}(\mathtt{b}, \mathtt{m})$-structure, but $A/\mathfrak{p}_3$ may not, although the latter can be canonically regarded as an $\mathscr{L}(\mathtt{m})$-structure); the fact that the residue domains $A/\mathfrak{p}_i$ may not be all canonically regarded as structures in the same language in the way described above creates an inherent difficulty in the model-theoretic analysis of local real closed SV-rings of finite rank in terms of the domains $A/\mathfrak{p}_i$ when there is more than one branching ideal.

There are also differences in terms of definability in the presence of more than one branching ideal. One fact about local real closed rings A of finite rank with one branching ideal $\mathfrak{b}_A$ that was used in the proof of completeness of $T_{n,2}$ (Corollary 4.5.27) is that $\mathfrak{b}_A$ is definable in A without parameters (Remark 4.5.8 (iii)); the next example shows that branching ideals are generally not definable without parameters in local real closed rings of finite rank which have more than one branching ideal:

Example 4.6.1. Let V be a real closed domain of Krull dimension 2 and let $\mathfrak{p}$ be a non-zero non-maximal prime ideal. Define $A_1 = A_2 := V \times_{V/\mathfrak{p}} V$, noting that A_1 is a local real closed ring of rank 2 with unique branching ideal $\ker(A_1 \twoheadrightarrow V/\mathfrak{p})$ and with residue field $V/\mathfrak{m}_V =: \boldsymbol{k}$. The ring $A := A_1 \times_{\boldsymbol{k}} A_2 \subseteq A_1 \times A_2$ is a local real closed ring of rank 4 with exactly 3 branching ideals, namely $\mathfrak{q}_1 := \ker(A \twoheadrightarrow A_1 \twoheadrightarrow V/\mathfrak{p})$, $\mathfrak{q}_2 := \ker(A \twoheadrightarrow A_2 \twoheadrightarrow V/\mathfrak{p})$, and $\mathfrak{m}_A = \ker(A \twoheadrightarrow \boldsymbol{k})$; if $\mathrm{Spec}^{\min}(A) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_4\}$, then the Hasse diagram of $(\mathrm{Spec}(A), \subseteq)$ is



and the map $(a_1, a_2) \mapsto (a_2, a_1)$ is an automorphism of $A \subseteq A_1 \times A_2$ which swaps $\mathfrak{q}_1$ and $\mathfrak{q}_2$.

Finally, recall that given a real closed valuation ring C, there always exists a non-trivial real closed valuation ring W with homomorphic image C yielding the homogeneous ring $\prod_C^n W$ of type (n, j) (if C is a field, then $j = 1$, otherwise $j = 2$), see Definition 4.4.19; loosely speaking, for any fixed "cofactor" C one can find a "factor" W yielding a ring of type $(n, 1)$ or of type $(n, 2)$ having the property that all its residue domains modulo minimal prime ideals are abstractly isomorphic to W. The model-theoretic relevance of this construction is that one can make the property of a tuple $\overline{a} \in W^n$ being an element of $\prod_C^n W$ "internal to W", in the sense that $\overline{a} \in \prod_C^n W$ if and only if $a_i - a_j \in \ker(W \twoheadrightarrow C)$ for all $i, j \in [n]$; this was crucially used in the model completeness proof of $T_{n,1}$, see Lemma 4.5.13 (II). Although it is easy to see that this construction can be done for all local real closed SV-rings of rank 3, the next example shows that this is not any more the case in the presence of two incomparable branching ideals (the precise statement is Claim 2 in the example below):

Example 4.6.2. Let $\boldsymbol{k}$ be a real closed field and Γ be a divisible totally ordered abelian group without a smallest non-zero convex subgroup (for instance, Γ can be taken to be $\mathbb{Q}^{\mathbb{N}}$ ordered lexicographically); in particular, $V_1 := \boldsymbol{k}[[\Gamma]]$ is a non-trivial real closed valuation ring without a largest non-maximal prime ideal, i.e., $\mathfrak{m}_{V_1}$ does not have an immediate predecessor in $(\mathrm{Spec}(V_1), \subseteq)$. Let $V_2 := \boldsymbol{k}[[\mathbb{Q}]]$, noting that $\mathrm{Spec}(V_2) = \{(0), \mathfrak{m}_{V_2}\}$.

Claim 1. There does not exist a surjective ring homomorphism $f : V_1 \twoheadrightarrow V_2$ nor a surjective ring homomorphism $g : V_2 \twoheadrightarrow V_1$.

Proof of Claim 1. Assume for contradiction that either f or g as in the statement of the claim exist. Note that since f and g are surjective, either $\mathrm{Spec}(V_2)$ is a final segment in $(\mathrm{Spec}(V_1), \subseteq)$ or $\mathrm{Spec}(V_1)$ is a final segment in $(\mathrm{Spec}(V_2), \subseteq)$, respectively; but this is impossible by choice of V_1 and V_2. $\square_{\text{Claim 1}}$

Claim 2. There does not exist a non-trivial real closed valuation ring W having both V_1 and V_2 as homomorphic images; in particular, there is no local real closed SV-ring of rank 4 of the form $(W \times_{V_1} W) \times_{\boldsymbol{k}} (W \times_{V_2} W)$.

Proof of Claim 2. Assume for contradiction that there exist surjective ring homomorphisms $f_1 : W \twoheadrightarrow V_1$ and $f_2 : W \twoheadrightarrow V_2$; since W is a valuation ring, either $\ker(f_1) \subseteq \ker(f_2)$ or $\ker(f_2) \subseteq \ker(f_1)$, therefore either V_2 is a homomorphic image of V_1 or V_1 is a homomorphic image of V_2, a contradiction to Claim 1. $\square_{\text{Claim 2}}$

4.6.2 The branching spectrum

Although individual branching ideals in an arbitrary local real closed ring A of finite rank are generally not definable without parameters (Example 4.6.1), the theory of A "knows" about the poset configuration of its branching ideals in a sense which is made precise in Corollary 4.6.8. First, some preliminaries are needed.

Definition 4.6.3. Let A be a local real closed ring of finite rank. Define the *branching spectrum of* A to be

$$\mathrm{BrSpec}(A) := \mathrm{Spec}^{\min}(A) \ \cup \ \{\mathfrak{m}_A\} \ \cup \ \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \text{ is a branching ideal}\}.$$

Definition 4.6.4. Let $(P, \sqsubseteq)$ be a *root system*, i.e., $(P, \sqsubseteq)$ is a poset such that the principal up-set $p^{\uparrow} := \{q \in P \mid p \sqsubseteq q\}$ is a chain for all $p \in P$.

(I) $q \in P$ is a *branching point* if there exist $p_1, p_2 \in P$ distinct from q such that $p_1, p_2 \sqsubseteq q$, and $\{q\} = (p_1^{\uparrow} \cap p_2^{\uparrow})^{\min}$ [4].

(II) P is a *root* if there exists an element $\top \in P$ such that $p \sqsubseteq \top$ for all $p \in P$ (note that if such element exists, it must be unique).

[4] If $S \subseteq (P, \sqsubseteq)$ is any subset, define $S^{\min} := \{s \in S \mid s \text{ is minimal in } S \text{ with respect to } \sqsubseteq\}$.

(III) Suppose that P is a finite root (i.e., P is a finite root system with unique maximal element).

(i) The *rank of* P is $\mathrm{rk}(P) := |P^{\min}|$.

(ii) The *branching root of* P is the subposet

$$\mathrm{Br}(P) := P^{\min} \ \cup\ \{\top\} \ \cup\ \{p \in P \mid p \text{ is a branching point}\}.$$

(iii) P is *reduced* if $P = \mathrm{Br}(P)$, i.e., P is reduced if every $p \in P \setminus (P^{\min} \cup \{\top\})$ is a branching point.

Remark 4.6.5. (i) Any finite root is a $\vee$-semilattice with join operation given by $p_1 \vee p_2 := (p_1^{\uparrow} \cap p_2^{\uparrow})^{\min}$.

(ii) By Remark 4.3.7, local real closed rings of finite rank have finitely many branching ideals; in particular, the poset $(\mathrm{BrSpec}(A), \subseteq)$ is a finite reduced root.

If $(P, \sqsubseteq)$ is a finite root system, then there exists a real closed ring A such that $(\mathrm{Spec}(A), \subseteq) \cong (P, \sqsubseteq)$. Indeed, by [DG00] there exists a ring B such that $(\mathrm{Sper}(B), \subseteq) \cong (P, \sqsubseteq)$, where $\mathrm{Sper}(B)$ is the real spectrum of B, see Subsection 2.2.1; then $(\mathrm{Spec}(A), \subseteq) \cong (P, \sqsubseteq)$, where $A := \rho(B)$ is the real closure of B, see [DST19, Section 13.6.3] and the references therein. The next lemma shows that one can in fact choose A to be a real closed SV-ring:

Lemma 4.6.6. *Let* $(P, \sqsubseteq)$ *be a finite root system. There exists a real closed SV-ring* A *such that* $(\mathrm{Spec}(A), \subseteq) \cong (P, \sqsubseteq)$.

Proof. First note that it suffices to prove the statement for finite roots. Indeed, if $(P, \sqsubseteq)$ is any finite root system, then $P = P_1 \,\dot{\cup}\, \ldots \,\dot{\cup}\, P_m$, where each $(P_i, \sqsubseteq)$ is a finite root; if A_i is a real closed SV-ring such that $(\mathrm{Spec}(A_i), \subseteq) \cong (P_i, \sqsubseteq)$ for all $i \in [m]$, then $A := A_1 \times \cdots \times A_m$ is a real closed SV-ring (by Proposition 4.2.4 (i) and Theorem 2.3.2 (I)) such that $(\mathrm{Spec}(A), \subseteq) \cong (P, \sqsubseteq)$.

The proof is now by induction on the rank of finite roots $(P, \sqsubseteq)$. If $\mathrm{rk}(P) = 1$, then $(P, \sqsubseteq)$ is a chain of n elements for some $n \in \mathbb{N}$, therefore choosing A to be a real closed valuation ring of Krull dimension $n-1$ does the job. Let $k \in \mathbb{N}$ and assume that the statement holds for all finite roots of rank k. Let $(P, \sqsubseteq)$ be a finite root of rank $k+1$ with minimal elements $p_1, \ldots, p_k, p_{k+1}$. Choose $i \in [k+1]$ such that $p_i^{\uparrow}$ is a chain of maximal cardinality $n \in \mathbb{N}$ in $(P, \sqsubseteq)$, pick any $j \in [k+1] \setminus \{i\}$, and define $P' := \bigcup_{\ell \in [k+1] \setminus \{j\}} p_\ell^{\uparrow} \subseteq P$; then $(P', \sqsubseteq)$ is a finite root of rank k, and thus by inductive hypothesis there exists a real closed SV-ring A' and a poset isomorphism $f : (P', \sqsubseteq) \longrightarrow (\mathrm{Spec}(A'), \subseteq)$. Define $V := A'/f(p_i)$ (noting that V is a real closed valuation ring of Krull dimension $n-1$) and $q := p_i \vee p_j$ (Remark 4.6.5 (i)); then $|p_j^{\uparrow}| := m \leq n$ by assumption on $p_i^{\uparrow}$, and this yields two possible cases:

- $m = n$. In this case, $(\mathrm{Spec}(A), \subseteq) \cong (P, \sqsubseteq)$ for $A := A' \times_{B/f(q)} B$, see [DST19, Section 12.5.7].

- $m < n$. In this case, $(\mathrm{Spec}(A), \subseteq) \cong (P, \sqsubseteq)$ for $A := A' \times_{B/f(q)} B/f(r)$, where $r \in p_i^{\uparrow}$ is such that $|r^{\uparrow}| = m$.

In each of the cases above, A is a real closed SV-ring by Proposition 4.2.4 (iv) and Theorem 2.3.2 (I); this concludes the inductive step and thus the proof. □

Lemma 4.6.7. *Let $(P, \sqsubseteq)$ be a finite reduced root of rank at least 2. There exists an $\mathscr{L}$-sentence $\varphi_{(P,\sqsubseteq)}$ such that*

$$A \models \varphi_{(P,\sqsubseteq)} \iff (\mathrm{BrSpec}(A), \subseteq) \cong (P, \sqsubseteq)$$

for all local real closed rings A of finite rank.

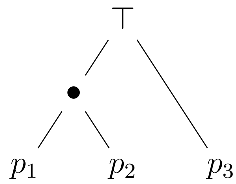
Proof. This is clear from combining Lemma 4.5.1 together with the following facts about a local real closed ring A of rank $n \in \mathbb{N}^{\geq 2}$:

(i) $\mathrm{Spec}^{\min}(A) = \{\mathrm{Ann}(a_i) \mid i \in [n]\}$ for all non-zero pairwise orthogonal elements $a_1, \ldots, a_n \in A$ (Lemma 4.2.9 (III) (ii));

(ii) each branching ideal of A is a sum of two distinct minimal prime ideals (Remark 4.3.6); and

(iii) the maximal ideal is a branching ideal if and only if every non-unit is a sum of two zero divisors (Proposition 4.3.5).

More precisely, assume without loss of generality that $(P, \sqsubseteq)$ is a finite reduced root of rank $n \in \mathbb{N}^{\geq 2}$ such that $\top$ is a branching point. Then $\varphi_{(P,\sqsubseteq)}$ can be taken to be the conjunction of: $\varphi_{\mathrm{rk}=n}$ (Lemma 4.5.1), the $\mathscr{L}$-sentence expressing "every non-unit is a sum of two zero divisors", and the $\mathscr{L}$-sentence expressing "there exist non-zero orthogonal elements $a_1, \ldots, a_n \in A$ such that $(\{\mathrm{Ann}(a_i) \mid i \in [n]\} \cup \{\mathrm{Ann}(a_i) + \mathrm{Ann}(a_j) \mid i, j \in [n]\}, \subseteq)$ is poset-isomorphic to $(P, \sqsubseteq)$". For instance, if $(P, \sqsubseteq)$ is the finite reduced root



then the last $\mathscr{L}$-sentence described above would be the one expressing "there exist non-zero pairwise orthogonal elements $a_1, a_2, a_3 \in A$ such that $\mathrm{Ann}(a_1) + \mathrm{Ann}(a_3) = \mathrm{Ann}(a_2) + \mathrm{Ann}(a_3)$ and $\mathrm{Ann}(a_1) + \mathrm{Ann}(a_2) \subsetneq \mathrm{Ann}(a_2) + \mathrm{Ann}(a_3)$". □

Corollary 4.6.8. *Let A and B be local real closed rings of finite rank. If $A \equiv B$, then* $(\mathrm{BrSpec}(A), \subseteq) \cong (\mathrm{BrSpec}(B), \subseteq)$.

Proof. Immediate from Lemma 4.6.7. □

Let A and B are local real closed SV-rings of finite rank $n \in \mathbb{N}^{\geq 2}$ with one branching ideal and suppose that $(\mathrm{BrSpec}(A), \subseteq) \cong (\mathrm{BrSpec}(B), \subseteq)$; then A is of type (n, j) $(j \in [2])$ if and only if B is of type (n, j), therefore $A \equiv B$ by Corollary 4.5.27. This observation gives rise to the following conjecture on an elementary classification of local real closed SV-rings of finite rank:

Conjecture 4.6.9. *Let A and B be local real closed SV-rings of finite rank. Then $A \equiv B$ if and only if* $(\mathrm{BrSpec}(A), \subseteq) \cong (\mathrm{BrSpec}(B), \subseteq)$.

If A is a local real closed ring of finite rank, then $\mathrm{BrSpec}(A)$ is a finite subset of the spectral space $\mathrm{Spec}(A)$, and as such, $\mathrm{BrSpec}(A)$ is proconstructible in $\mathrm{Spec}(A)$ (i.e., it is a spectral subspace of $\mathrm{Spec}(A)$, see Section 2.2); in particular, A corresponds to the *radical relation* $\preceq_{\mathrm{BrSpec}(A)} \subseteq A^2$ on A. Radical relations on rings are certain binary relations which were introduced in [PS90] and later used in [PS] for the model-theoretic analysis of real closed rings. It is shown in [PS] that if A is any real closed ring, then

$$X \subseteq \mathrm{Spec}(A) \longmapsto a \preceq_X b \overset{\mathrm{def}}{\Longleftrightarrow} \forall \mathfrak{p} \in X[b \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}]$$

is a bijection between proconstructible subsets $X \subseteq \mathrm{Spec}(A)$ and radical relations $\preceq$ on A, therefore a real closed ring with a radical relation $(A, \preceq_X)$ "knows" about the spectral space X since the bounded and distributive lattice $\overline{\mathcal{K}}(X)$ of closed constructible subsets of X is interpretable in $(A, \preceq_X)$; furthermore, the model theory of real closed valuation rings with radical relations is well-understood from the work carried in the last three sections of [PS]. In view of all of the above, a possible approach to a uniform model-theoretic analysis of all local real closed SV-rings of finite rank and to answer Conjecture 4.5.30 in the affirmative is to study such rings equipped with the radical relation corresponding to their branching spectrum.

Bibliography

[ADH17] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven. *Asymptotic differential algebra and model theory of transseries*. Vol. 195. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2017, pp. xxi+849 (cit. on pp. 15, 39, 180, 181).

[AF88] Marlow Anderson and Todd Feil. *Lattice-ordered groups*. Reidel Texts in the Mathematical Sciences. An introduction. D. Reidel Publishing Co., Dordrecht, 1988, pp. x+190 (cit. on pp. 54, 55, 56).

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969, pp. ix+128 (cit. on p. 12).

[Ast08] Vincent Astier. "Elementary equivalence of some rings of definable functions". In: *Arch. Math. Logic* 47.4 (2008), pp. 327–340 (cit. on p. 10).

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*. Vol. 36. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Translated from the 1987 French original, Revised by the authors. Springer-Verlag, Berlin, 1998, pp. x+430 (cit. on pp. 12, 41, 42, 43, 45, 47, 48, 50).

[Bec83] Thomas Becker. "Real closed rings and ordered valuation ring". In: *Mathematical Logic Quarterly* 29.8 (1983), pp. 417–425 (cit. on p. 118).

[BKW77] Alain Bigard, Klaus Keimel, and Samuel Wolfenstein. *Groupes et anneaux réticulés*. Vol. 608. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1977, pp. xi+334 (cit. on pp. 54, 55, 56, 68, 117, 160, 161).

[BW79] Stanley Burris and Heinrich Werner. "Sheaf constructions and their elementary properties". In: *Trans. Amer. Math. Soc.* 248.2 (1979), pp. 269–309 (cit. on pp. 122, 123).

[CD83] Gregory Cherlin and Max A. Dickmann. "Real closed rings. II. Model theory". In: *Ann. Pure Appl. Logic* 25.3 (1983), pp. 213–231 (cit. on pp. 10, 38, 40, 41, 117, 118).

[CD86] Gregory L. Cherlin and Max A. Dickmann. "Real closed rings. I. Residue rings of rings of continuous functions". In: *Fund. Math.* 126.2 (1986), pp. 147–183 (cit. on p. 117).

[CGL99] R. Cignoli, D. Gluschankof, and F. Lucas. "Prime spectra of lattice-ordered abelian groups". In: *J. Pure Appl. Algebra* 136.3 (1999), pp. 217–229 (cit. on p. 59).

[Che80] Gregory Cherlin. "Rings of continuous functions: decision problems". In: *Model theory of algebra and arithmetic (Proc. Conf., Karpacz, 1979).* Vol. 834. Lecture Notes in Math. Springer, Berlin, 1980, pp. 44–91 (cit. on p. 10).

[CK90] C. C. Chang and H. J. Keisler. *Model theory.* Third. Vol. 73. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1990 (cit. on pp. 12, 35, 41, 57, 155, 157, 159).

[CMS21] Jennifer Chubb, Russell Miller, and Reed Solomon. "Model completeness and relative decidability". In: *Arch. Math. Logic* 60.6 (2021), pp. 721–735 (cit. on p. 26).

[CR82] Michel Coste and Marie-Françoise Roy. "La topologie du spectre réel". In: *Ordered fields and real algebraic geometry (San Francisco, Calif., 1981).* Vol. 8. Contemp. Math. Amer. Math. Soc., Providence, RI, 1982, pp. 27–59 (cit. on p. 30).

[Dar95] Michael R. Darnel. *Theory of lattice-ordered groups.* Vol. 187. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1995, pp. viii+539 (cit. on pp. 54, 55, 56).

[DG00] M. Dickmann and D. Gluschankof. “The order structure of the real spectrum of commutative rings”. In: *J. Algebra* 229.1 (2000), pp. 175–204 (cit. on p. 166).

[dHJ24] Christian d’Elbée, Yatir Halevi, and Will Johnson. *The classification of dp-minimal integral domains.* 2024. arXiv: `2405.21014 [math.LO]`. (Visited on 12/07/2024) (cit. on p. 158).

[Dic87] M. A. Dickmann. “Elimination of quantifiers for ordered valuation rings”. In: *J. Symbolic Logic* 52.1 (1987), pp. 116–128 (cit. on p. 158).

[DL95] Lou van den Dries and Adam H. Lewenberg. “T-convexity and tame extensions”. In: *J. Symbolic Logic* 60.1 (1995), pp. 74–102 (cit. on pp. 42, 43, 44, 114).

[DM95] Charles Delzell and James Madden. “Lattice-ordered rings and semialgebraic geometry. I”. In: *Real analytic and algebraic geometry (Trento, 1992).* de Gruyter, Berlin, 1995, pp. 103–129 (cit. on pp. 35, 114, 160).

[Dri97] Lou van den Dries. “T-convexity and tame extensions. II”. In: *J. Symbolic Logic* 62.1 (1997), pp. 14–34 (cit. on p. 184).

[Dri98] Lou van den Dries. *Tame topology and o-minimal structures.* Vol. 248. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1998, pp. x+180 (cit. on pp. 12, 42, 43, 47, 49, 66, 114).

[DST19] Max Dickmann, Niels Schwartz, and Marcus Tressl. *Spectral spaces.* Vol. 35. New Mathematical Monographs. Cambridge University Press, Cambridge, 2019, pp. xvii+633 (cit. on pp. 27, 28, 29, 30, 32, 34, 136, 141, 166, 167).

[DT20] Luck Darnière and Marcus Tressl. “Defining integer-valued functions in rings of continuous definable functions over a topological field”. In: *J. Math. Log.* 20.3 (2020), pp. 2050014, 24 (cit. on p. 11).

[DW96] H. Garth Dales and W. Hugh Woodin. *Super-real fields.* Vol. 14. London Mathematical Society Monographs. New Series. Totally ordered fields with additional structure, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996, pp. xvi+357 (cit. on p. 39).

[EG98] Yu. L. Ershov and S. S. Goncharov. "Elementary theories and their constructive models". In: *Handbook of recursive mathematics, Vol. 1*. Vol. 138. Stud. Logic Found. Math. North-Holland, Amsterdam, 1998, pp. 115–165 (cit. on pp. 25, 26).

[EP05] Antonio Engler and Alexander Prestel. *Valued fields*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005, pp. x+205 (cit. on pp. 37, 39, 180, 185).

[EP84] Yu. L. Ershov and E. A. Palyutin. *Mathematical logic*. Translated from the Russian by Vladimir Shokurov. "Mir", Moscow, 1984, p. 303 (cit. on p. 22).

[Ers+98] Yu. L. Ershov et al., eds. *Handbook of recursive mathematics. Vol. 1*. Vol. 138. Studies in Logic and the Foundations of Mathematics. Recursive model theory. North-Holland, Amsterdam, 1998, pp. xlvi+620 (cit. on p. 25).

[Fuc70] L. Fuchs. *Infinite Abelian Groups. Volume 1*. Academic Press, 1970 (cit. on p. 182).

[FV59] S. Feferman and R. L. Vaught. "The first order properties of products of algebraic systems". In: *Fund. Math.* 47 (1959), pp. 57–103 (cit. on p. 10).

[GG24] Mai Gehrke and Sam van Gool. *Topological Duality for Distributive Lattices: Theory and Applications*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2024 (cit. on p. 27).

[GJ60] Leonard Gillman and Meyer Jerison. *Rings of continuous functions*. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960, pp. ix+300 (cit. on p. 117).

[Gui01] Jorge I. Guier. "Boolean products of real closed valuation rings and fields". In: *Ann. Pure Appl. Logic* 112.2-3 (2001), pp. 119–150 (cit. on p. 122).

[Gui25] Jorge I. Guier. "Elimination of quantifiers for a theory of real closed rings". In: *Ann. Pure Appl. Logic* 176.1 (2025) (cit. on p. 162).

[Har98] Valentina S. Harizanov. "Pure computable model theory". In: *Handbook of recursive mathematics, Vol. 1.* Vol. 138. Stud. Logic Found. Math. North-Holland, Amsterdam, 1998, pp. 3–114 (cit. on pp. 15, 21).

[Hen+94] Melvin Henriksen et al. "Lattice-ordered algebras that are subdirect products of valuation domains". In: *Trans. Amer. Math. Soc.* 345.1 (1994), pp. 195–221 (cit. on p. 117).

[Her69] Hans Hermes. *Enumerability, decidability, computability. An introduction to the theory of recursive functions.* revised. Vol. Band 127. Die Grundlehren der mathematischen Wissenschaften. Translated from the German by G. T. Hermann and O. Plassmann. Springer-Verlag New York, Inc., New York, 1969, pp. xi+245 (cit. on p. 21).

[HL93] Melvin Henriksen and Suzanne Larson. "Semiprime f-rings that are subdirect products of valuation domains". In: *Ordered algebraic structures (Gainesville, FL, 1991).* Kluwer Acad. Publ., Dordrecht, 1993, pp. 159–168 (cit. on p. 117).

[Hoc69] M. Hochster. "Prime ideal structure in commutative rings". In: *Trans. Amer. Math. Soc.* 142 (1969), pp. 43–60 (cit. on p. 29).

[Hod93] Wilfrid Hodges. *Model theory.* Vol. 42. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1993, pp. xiv+772 (cit. on pp. 12, 158).

[HW92] Melvin Henriksen and Richard Wilson. "When is $C(X)/P$ a valuation ring for every prime ideal P?" In: *Proceedings of the Symposium on General Topology and Applications (Oxford, 1989).* Vol. 44. 1-3. 1992, pp. 175–180 (cit. on pp. 117, 121).

[Joh82] Peter T. Johnstone. *Stone spaces.* Vol. 3. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1982, pp. xxi+370 (cit. on pp. 27, 121).

[Kap42] Irving Kaplansky. "Maximal fields with valuations". In: *Duke Math. J.* 9 (1942), pp. 303–321 (cit. on p. 184).

[KS22] Manfred Knebusch and Claus Scheiderer. *Real Algebra: An Introduction*. Springer, 2022, pp. XII–206 (cit. on pp. 37, 108, 181, 183).

[KS99] Bakhadyr Khoussainov and Richard A. Shore. "Effective model theory: the number of models and their complexity". In: *Models and computability (Leeds, 1997)*. Vol. 259. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1999, pp. 193–239 (cit. on pp. 24, 25).

[Lar10] Suzanne Larson. "SV and related f-rings and spaces". In: *Ann. Fac. Sci. Toulouse Math. (6)* 19.Fascicule Spécial (2010), pp. 111–141 (cit. on pp. 117, 124, 140).

[Lar11] Suzanne Larson. "Finitely 1-convex f-rings". In: *Topology Appl.* 158.14 (2011), pp. 1888–1901 (cit. on pp. 117, 127, 140, 145).

[Lev09] Oscar L. Levin. "Computability Theory, Reverse Mathematics, and Ordered Fields". Ph.D. thesis. University of Connecticut, 2009 (cit. on p. 25).

[Mac98] Saunders Mac Lane. *Categories for the working mathematician*. Second. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998, pp. xii+314 (cit. on p. 12).

[Mad70] E. W. Madison. "A note on computable real fields". In: *J. Symbolic Logic* 35 (1970), pp. 239–241 (cit. on p. 25).

[Man10] Yu. I. Manin. *A course in mathematical logic for mathematicians*. Second. Vol. 53. Graduate Texts in Mathematics. Chapters I–VIII translated from the Russian by Neal Koblitz, With new chapters by Boris Zilber and the author. Springer, New York, 2010, pp. xviii+384 (cit. on pp. 21, 22).

[Mar02] David Marker. *Model theory*. Vol. 217. Graduate Texts in Mathematics. An introduction. Springer-Verlag, New York, 2002, pp. viii+342 (cit. on pp. 12, 57, 81).

[Mat83] Eben Matlis. "The minimal prime spectrum of a reduced ring". In: *Illinois J. Math.* 27 (1983) (cit. on p. 125).

[Men62] Bert Mendelson. *Introduction to topology*. College Mathematics Series. Allyn and Bacon, Inc., Boston, MA, 1962, pp. ix+217 (cit. on p. 50).

[Mil99] Terrence S. Millar. "Pure recursive model theory". In: *Handbook of computability theory*. Vol. 140. Stud. Logic Found. Math. North-Holland, Amsterdam, 1999, pp. 507–532 (cit. on p. 21).

[MMS00] Dugald Macpherson, David Marker, and Charles Steinhorn. "Weakly o-minimal structures and real closed fields". In: *Trans. Amer. Math. Soc.* 352.12 (2000), pp. 5435–5483 (cit. on pp. 117, 118).

[Mon21] Antonio Montalbán. *Computable structure theory—within the arithmetic*. Perspectives in Logic. Cambridge University Press, Cambridge; Association for Symbolic Logic, Ithaca, NY, 2021, pp. xxii+190. ISBN: 978-1-108-42329-8 (cit. on pp. 24, 25, 112).

[MR89] H. Matsumura and M. Reid. *Commutative Ring Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1989 (cit. on p. 12).

[Mur99] Roman Murawski. *Recursive functions and metamathematics*. Vol. 286. Synthese Library. Problems of completeness and decidability, Gödel's theorems. Kluwer Academic Publishers Group, Dordrecht, 1999, pp. xii+391 (cit. on pp. 21, 113).

[Phi15] Laura Rose Phillips. "Some structures interpretable in the ring of continuous semi-algebraic functions on a curve". Ph.D. thesis. University of Manchester, 2015 (cit. on p. 48).

[Pri83] Sibylla Prieß-Crampe. *Angeordnete Strukturen*. Vol. 98. Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]. Gruppen, Körper, projektive Ebenen. [Groups, fields, projective planes]. Springer-Verlag, Berlin, 1983 (cit. on pp. 39, 183).

[Pru01] Mihai Prunescu. "Non-effective quantifier elimination". In: *MLQ Math. Log. Q.* 47.4 (2001), pp. 557–561 (cit. on p. 23).

[PS] Alexander Prestel and Niels Schwartz. "Model theory of real closed rings". In: *Valuation theory and its applications, Vol. I (Saskatoon, SK, 1999)*. Vol. 32. Fields Inst. Commun. Pp. 261–290 (cit. on pp. 33, 36, 120, 148, 149, 155, 159, 162, 168, 169).

[PS86] Anand Pillay and Charles Steinhorn. "Definable sets in ordered structures. I". In: *Trans. Amer. Math. Soc.* 295.2 (1986), pp. 565–592 (cit. on pp. 12, 42, 185).

[PS90] Alexander Prestel and James Schmid. "Existentially closed domains with radical relations: an axiomatization of the ring of algebraic integers". In: *J. Reine Angew. Math.* 407 (1990), pp. 178–201 (cit. on p. 168).

[Rab60] Michael O. Rabin. "Computable algebra, general theory and theory of computable fields". In: *Trans. Amer. Math. Soc.* 95 (1960), pp. 341–360 (cit. on p. 26).

[Rab69] Michael O. Rabin. "Decidability of second-order theories and automata on infinite trees". In: *Trans. Amer. Math. Soc.* 141 (1969), pp. 1–35 (cit. on p. 112).

[Rab77] Michael O. Rabin. "Decidable theories". In: *Handbook of mathematical logic*. Vol. 90. Stud. Logic Found. Math. North-Holland, Amsterdam, 1977, pp. 595–629 (cit. on p. 15).

[Rid14] Silvain Rideau-Kikuchi. "Éliminations dans les corps valués". Ph.D. thesis. Université Paris-Sud, 2014 (cit. on pp. 15, 16, 18).

[Rog67] Hartley Rogers Jr. *Theory of recursive functions and effective computability*. McGraw-Hill Book Co., New York-Toronto-London, 1967, pp. xx+482 (cit. on p. 21).

[Sch09] Niels Schwartz. "Real closed valuation rings". In: *Comm. Algebra* 37.11 (2009), pp. 3796–3814 (cit. on pp. 117, 183).

[Sch10a] Niels Schwartz. "Convex subrings of partially ordered rings". In: *Math. Nachr.* 283.5 (2010), pp. 758–774 (cit. on p. 160).

[Sch10b] Niels Schwartz. "SV-rings and SV-porings". In: *Ann. Fac. Sci. Toulouse Math. (6)* 19.Fascicule Spécial (2010), pp. 159–202 (cit. on pp. 117, 118, 121, 122, 123, 124, 126, 160).

[Sch13] Niels Schwartz. "Sheaves of abelian l-groups". In: *Order* 30.2 (2013), pp. 497–526 (cit. on pp. 58, 59, 68).

[Sch86] Niels Schwartz. “Real closed rings”. In: *Algebra and order (Luminy-Marseille, 1984)*. Vol. 14. Res. Exp. Math. Heldermann, Berlin, 1986, pp. 175–194 (cit. on pp. 33, 34, 117).

[Sch87] Niels Schwartz. *The basic theory of real closed spaces*. Vol. 15. Regensburger Mathematische Schriften [Regensburg Mathematical Publications]. Universität Regensburg, Fachbereich Mathematik, Regensburg, 1987, pp. xiv+257 (cit. on p. 9).

[Sch89] Niels Schwartz. “The basic theory of real closed spaces”. In: *Mem. Amer. Math. Soc.* 77.397 (1989), pp. viii+122 (cit. on pp. 9, 32, 33, 34, 36, 75, 117).

[Sch91] Niels Schwartz. “Inverse real closed spaces”. In: *Illinois J. Math.* 35.4 (1991), pp. 536–568 (cit. on p. 10).

[Sch97] Niels Schwartz. “Rings of continuous functions as real closed rings”. In: *Ordered algebraic structures (Curaçao, 1995)*. Kluwer Acad. Publ., Dordrecht, 1997, pp. 277–313 (cit. on pp. 33, 117).

[Sim15] Pierre Simon. *A guide to NIP theories*. Vol. 44. Lecture Notes in Logic. Association for Symbolic Logic, Chicago, IL; Cambridge Scientific Publishers, Cambridge, 2015, pp. vii+156 (cit. on pp. 118, 158).

[SM99] Niels Schwartz and James J. Madden. *Semi-algebraic function rings and reflectors of partially ordered rings*. Vol. 1712. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1999, pp. xii+279 (cit. on pp. 33, 34, 36, 117, 140, 160, 161).

[ST10] Niels Schwartz and Marcus Tressl. “Elementary properties of minimal and maximal points in Zariski spectra”. In: *J. Algebra* 323.3 (2010), pp. 698–728 (cit. on p. 121).

[ST99] V. Stoltenberg-Hansen and J. V. Tucker. “Computable rings and fields”. In: *Handbook of computability theory*. Vol. 140. Stud. Logic Found. Math. North-Holland, Amsterdam, 1999, pp. 363–447 (cit. on p. 26).

[Ste10] Stuart A. Steinberg. *Lattice-ordered rings and modules*. Springer, New York, 2010, pp. xviii+630 (cit. on pp. 78, 85).

[SW87a] Fuxing Shen and Volker Weispfenning. *Continuous functions: ℓ-groups and decision problems*. 1987 (cit. on pp. 19, 53, 59, 71).

[SW87b] Fuxing Shen and Volker Weispfenning. *Continuous functions: ℓ-groups and decision problems*. 1987 (cit. on p. 59).

[Tre] Marcus Tressl. *The Shen-Weispfenning paper*. Private notes (cit. on p. 54).

[Tre07] Marcus Tressl. "Super real closed rings". In: *Fund. Math.* 194.2 (2007), pp. 121–177 (cit. on p. 117).

[Tre09] Marcus Tressl. "Heirs of box types in polynomially bounded structures". In: *J. Symbolic Logic* 74.4 (2009), pp. 1225–1263 (cit. on p. 40).

[Tre16] Marcus Tressl. *Interpreting formulas of divisible lattice ordered abelian groups*. 2016. arXiv: 1609.07522 [math.LO]. (Visited on 10/12/2024) (cit. on pp. 74, 112).

[Tre17] Marcus Tressl. "On the strength of some topological lattices". In: *Ordered algebraic structures and related topics*. Vol. 697. Contemp. Math. Amer. Math. Soc., Providence, RI, 2017, pp. 325–347 (cit. on pp. 74, 94, 112).

[Tre22] Marcus Tressl. *Lattice-Ordered Abelian Groups and Their Model Theory*. Lecture notes from a course given at Università degli Studi della Campania Luigi Vanvitelli at Caserta. 2022 (cit. on pp. 54, 67).

[Wei75] Volker Weispfenning. "Model-completeness and elimination of quantifiers for subdirect products of structures". In: *J. Algebra* 36.2 (1975), pp. 252–277 (cit. on p. 121).

[Wei78] Volker Weispfenning. "Model Theory of Lattice Products". Habilitation thesis. Heidelberg University, 1978 (cit. on p. 121).

[Wil70] Stephen Willard. *General topology*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970, pp. xii+369 (cit. on p. 66).

Appendix A

Embedding Real Closed Valued Fields

The aim of this section is proving Theorem A.5. Familiarity with the basic notions and properties of valued fields is assumed throughout (see for example [EP05, Chapter 2] or [ADH17, Chapter 3]), as well as familiarity with ordered and real closed fields; in what follows, the notation and conventions used for this appendix are fixed.

A *valuation* on a field K is a function $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$, where Γ is a totally ordered abelian group and ∞ is a symbol satisfying $\gamma < \infty$ for all $\gamma \in \Gamma$, such that

(i) $v(a) = \infty$ if and only if $a = 0$,

(ii) $v(ab) = v(a) + v(b)$, and

(iii) $\min\{v(a), v(b)\} \leq v(a + b)$.

Every valuation on a field K is denoted by v, with the exception of the canonical valuation on fields of Hahn series $\boldsymbol{k}((\Gamma))$, in which case the valuation is denoted by ν, see Theorem 2.3.8; in particular, if $K \subseteq L$ is an extension of valued fields, then the valuation on L is v and the valuation on K is $v_{\restriction K}$. If K is a valued field, then $V_K := \{a \in K \mid v(a) \geq 0\}$ is the corresponding valuation ring and $\lambda_K : V \twoheadrightarrow V/\mathfrak{m}_K$ is the residue field map, where $\mathfrak{m}_K := \{a \in K \mid v(a) > 0\}$ is the unique maximal ideal of V_K; write $V := V_K$, $\mathfrak{m} := \mathfrak{m}_K$, and $\lambda := \lambda_K$ if K is clear from the context.

An *ordered valued field* is a totally ordered field K together with an *order-compatible valuation* (also called *convex valuation*) $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$, i.e., for all $a, b \in K$, if $0 < a < b$, then $v(b) \leq v(a)$. If K is a totally ordered field and $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$ is a valuation on K, then the following are equivalent:

(i) $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$ is an order-compatible valuation.

(ii) V is convex in K.

(iii) The composite map $K^{>0} \twoheadrightarrow \Gamma \twoheadrightarrow \Gamma^{\mathrm{op}}$ given by $a \mapsto -v(a)$ is a surjective morphism of totally ordered groups, where Γ^{op} denotes the totally ordered group obtained by reversing the order of Γ; in particular, $\ker(v_{\restriction K^{>0}})$ is a multiplicative convex subgroup of $K^{>0}$.

Since convex subrings of totally ordered fields are valuation rings ([KS22, Proposition 2.2.4]), ordered valued fields can be equivalently defined as pairs (K, V), where K is an ordered field and $V \subseteq K$ is a convex subring; in particular, if K is an ordered valued field, then its residue field $\boldsymbol{k} := V/\mathfrak{m}$ is endowed with a canonical total order turning it into a totally ordered field in such a way that the residue field map $\lambda : V \twoheadrightarrow \boldsymbol{k}$ is order-preserving.

A *real closed valued field* is an ordered valued field which is real closed as a field, i.e., it is a real closed field equipped with an order-compatible valuation; equivalently, it is a real closed field with a distinguished convex subring. If K is an ordered valued field, then its real closure $\rho(K)$ will be regarded as a real closed valued field with the valuation induced by K, i.e., $V_{\rho(K)}$ is defined as the convex hull of V_K in $\rho(K)$; if the value group and the residue field of K are Γ and $\boldsymbol{k}$ (respectively), then the value group and the residue field of $\rho(K)$ are $\mathbb{Q}\Gamma$ and $\rho(\boldsymbol{k})$ (respectively), and the field embedding $\rho_K : K \hookrightarrow \rho(K)$ is an embedding of valued fields, see [ADH17, Corollary 3.5.18]. Any isomorphism of ordered valued fields $K \longrightarrow L$ extends uniquely to an isomorphism of valued fields $\rho(K) \longrightarrow \rho(L)$, which is also order-preserving since $\rho(K)$ and $\rho(L)$ are real closed; therefore, if R is a real closed valued field and $\varepsilon : K \hookrightarrow R$ is an embedding of ordered valued fields, then ε can be extended uniquely to an embedding of valued fields $\rho(K) \hookrightarrow R$.

Lemma A.1. *Let K be a real closed valued field with value group Γ and $G \subseteq K^{>0}$ be a subgroup. The following are equivalent:*

(i) *G is a* monomial group *of K, i.e., $v_{\restriction G} : G \longrightarrow \Gamma$ is a group isomorphism.*

(ii) *G is a subgroup of $K^{>0}$ maximal with $G \cap \ker(v_{\restriction K^{>0}}) = (1)$.*

In particular:

(a) *Every real closed valued field has a monomial group.*

(b) *If $K \subseteq L$ is an extension of real closed valued fields and G is a monomial group of K, then there exists a monomial group H of L containing G.*

Proof. (i) $\Rightarrow$ (ii). Since $v_{\restriction G}$ is injective, $G \cap \ker(v_{\restriction K^{>0}}) = (1)$. Assume for contradiction that there exists a subgroup $G \subsetneq G' \subseteq K^{>0}$ with $G' \cap \ker(v_{\restriction G'}) = (1)$ and pick $g' \in G' \setminus G$; since $v_{\restriction G}$ is surjective, there exists $g \in G$ with $v(g) = v(g')$, hence $g'g^{-1} \in G' \cap \ker(v_{\restriction K^{>0}}) = (1)$, and thus $g' = g$, a contradiction to the choice of g'.

(ii) $\Rightarrow$ (i). Since v is an order-compatible valuation on K, $\ker(v_{\restriction K^{>0}})$ is a convex subgroup of $K^{>0}$; since K is real closed, $K^{>0}$ is multiplicatively divisible (i.e., it has nth roots for every positive integer n), and thus $\ker(v_{\restriction K^{>0}})$ is a divisible subgroup of $K^{>0}$. By [Fuc70, Theorem 21.2] and by assumption on G, $K^{>0} = \ker(v_{\restriction K^{>0}}) \cdot G$, i.e., $v_{\restriction G} : G \longrightarrow \Gamma$ is a group isomorphism, as required.
Items (a) and (b) follow from the implication (ii) $\Rightarrow$ (i) and an application of Zorn's lemma. □

Lemma A.2. *Let K be a real closed valued field with residue field $\boldsymbol{k}$ and $\boldsymbol{k}_0 \subseteq V$ be a subfield. The following are equivalent:*

(i) *$\boldsymbol{k}_0$ is a* coefficient field *of K, i.e., $\lambda_{\restriction \boldsymbol{k}_0} : \boldsymbol{k}_0 \longrightarrow \boldsymbol{k}$ is a field isomorphism.*

(ii) *$\boldsymbol{k}_0$ is a maximal subfield of V.*

In particular:

(a) *Every real closed valued field has a coefficient field.*

(b) *If $K \subseteq L$ is an extension of real closed valued fields and $\boldsymbol{k}_0$ is a coefficient field of K, then there exists a coefficient field $\boldsymbol{l}_0$ of L containing $\boldsymbol{k}_0$.*

Proof. (i) $\Rightarrow$ (ii). Assume for contradiction that there exists a subfield $\boldsymbol{k}_0 \subsetneq \boldsymbol{k}' \subseteq V$ and pick $a' \in \boldsymbol{k}' \setminus \boldsymbol{k}_0$; since $\lambda_{\restriction \boldsymbol{k}_0}$ is surjective, there exists $a \in \boldsymbol{k}_0$ with $\lambda(a) = \lambda(a')$, hence $a - a' \in \boldsymbol{k}' \cap \mathfrak{m} = (0)$, and thus $a' = a$, a contradiction to the choice of a'.

(ii) $\Rightarrow$ (i). Folklore; see for instance [KS22, Proposition 2.5.3] or [Sch09, Proposition 2.1].

Items (a) and (b) follow from the implication (ii) $\Rightarrow$ (i) and an application of Zorn's lemma. □

Lemma A.3. *Let K be an ordered valued field with value group Γ and residue field $\boldsymbol{k}$, and suppose that $G \subseteq K^{>0}$ is a monomial group of K. If $\varepsilon : K \hookrightarrow \boldsymbol{k}((\Gamma))$ is an embedding of valued fields such that $\varepsilon(g) = x^{v(g)}$ for all $g \in G$, then ε preserves the order.*

Proof. Let $r \in K^{>0}$, assume without loss of generality that $r \in V$ (otherwise replace r by r^{-1}), and write

$$\varepsilon(r) := a_{\gamma_0} x^{\gamma_0} + \sum a_\gamma x^\gamma,$$

where $\gamma_0 := \nu(\varepsilon(r)) = v(r) \in \Gamma$; it must be shown that $\varepsilon(r) > 0$, i.e., that $a_{\gamma_0} > 0$. Let $g \in G$ be such that $v(g) = \gamma_0$; then $0 = v(rg^{-1}) = \nu(\varepsilon(rg^{-1}))$, and

$$\varepsilon(rg^{-1}) = \varepsilon(r)\varepsilon(g^{-1}) = a_{\gamma_0} + \sum a_\gamma x^{\gamma-\gamma_0} \in \boldsymbol{k}[[\Gamma]],$$

therefore $0 \neq a_{\gamma_0} = \lambda_{\boldsymbol{k}((\Gamma))}(\varepsilon(rg^{-1})) = \lambda_K(rg^{-1})$, and since $g > 0$, $r > 0$, and $\lambda_K : V \twoheadrightarrow \boldsymbol{k}$ is order-preserving, $a_{\gamma_0} = \lambda_K(rg^{-1}) > 0$ follows, as required. □

Theorem A.4. *Let K be a real closed valued field with value group Γ and residue field $\boldsymbol{k}$. Suppose that $G \subseteq K^{>0}$ is a monomial group of K and $\boldsymbol{k}_0 \subseteq V$ is a coefficient field of K. There exists an embedding of valued fields $\varepsilon : K \hookrightarrow \boldsymbol{k}((\Gamma))$ such that $\varepsilon(g) = x^{v(g)}$ for all $g \in G$ and $\varepsilon(a) = \lambda(a)$ for all $a \in \boldsymbol{k}_0$.*

Proof. See [Pri83, Satz 21, p. 62]. □

Theorem A.5. *Let $K \subseteq L$ be an extension of real closed valued fields with value groups Γ and Δ, and residue fields $\boldsymbol{k}$ and $\boldsymbol{l}$, respectively. There exist embeddings of valued fields $\varepsilon_K : K \hookrightarrow \boldsymbol{k}((\Gamma))$ and $\varepsilon_L : L \hookrightarrow \boldsymbol{l}((\Delta))$ such that $\varepsilon_{L\restriction K} = \varepsilon_K$.*

Proof. Let $G \subseteq K^{>0}$ be a monomial group of K, $H \subseteq L^{>0}$ be a monomial group of L containing G, $\boldsymbol{k}_0 \subseteq V_K$ be a coefficient field of K, and $\boldsymbol{l}_0 \subseteq V_L$ be a coefficient field of L containing $\boldsymbol{k}_0$; these exist by items (a) and (b) in Lemmas A.1 and A.2.

Claim. $K^{>0} \cap H = G$ and $K \cap \boldsymbol{l}_0 = \boldsymbol{k}_0$.

Proof of Claim. Clearly $G \subseteq K^{>0} \cap H$ and $\boldsymbol{k}_0 \subseteq K \cap \boldsymbol{l}_0$. Since $K \subseteq L$ as valued fields, $\ker(v_{\restriction K^{>0}}) \subseteq \ker(v_{\restriction L^{>0}})$, and since H is a monomial group of L, $H \cap \ker(v_{\restriction L^{>0}}) = (1)$; therefore $H \cap \ker(v_{\restriction K^{>0}}) = (1)$, hence $(K^{>0} \cap H) \cap \ker(v_{\restriction K^{>0}}) = (1)$, and thus $K^{>0} \cap H = G$ by the implication (i) $\Rightarrow$ (ii) in Lemma A.1. Similarly, since $K \subseteq L$ as valued fields, $K \cap V_L = V_K$, hence $\boldsymbol{k}_0 \subseteq K \cap \boldsymbol{l}_0 \subseteq K \cap V_L = V_K$, and thus $\boldsymbol{k}_0 = K \cap \boldsymbol{l}_0$ by the implication (i) $\Rightarrow$ (ii) in Lemma A.2. $\square_{\text{Claim}}$

By Theorem A.4, there exists an embedding of valued fields $\varepsilon_K : K \longrightarrow \boldsymbol{k}((\Gamma))$ such that $\varepsilon(g) = x^{v(g)}$ for all $g \in G$ and $\varepsilon(a) = \lambda(a)$ for all $a \in \boldsymbol{k}_0$; the goal is to extend ε_K to an embedding of valued fields $\varepsilon_L : L \hookrightarrow \boldsymbol{l}((\Delta))$. If the extension $K \subseteq L$ is immediate (that is, if K and L have the same value groups and the same residue fields), then $G = H$, $\boldsymbol{k}_0 = \boldsymbol{l}_0$, and $\boldsymbol{k}((\Gamma)) = \boldsymbol{l}((\Delta))$, therefore the existence of an embedding of valued fields $\varepsilon_L : L \hookrightarrow \boldsymbol{l}((\Delta))$ such that $\varepsilon_{L\restriction K} = \varepsilon_K$ follows by [Kap42, Theorem 5]. Suppose now that the extension $K \subseteq L$ is not immediate; by induction it suffices to consider the case $L := K\langle r\rangle$, where $r \in L \setminus K$ and $K\langle r\rangle := \rho(K(r))$ (note that any such r is transcendental over K, as otherwise the field generated by $K \cup \{r\}$ in L is a proper real algebraic extension of K, contradicting the fact that K is real closed). By the Wilkie inequality ([Dri97, Corollary 5.6]), there are two cases to consider:

<u>Case 1.</u> $\boldsymbol{k} = \boldsymbol{l}$ (hence $\boldsymbol{k}_0 = \boldsymbol{l}_0$) and there exists $\delta \in \Delta \setminus \Gamma$ such that $\Delta = \Gamma \oplus \mathbb{Q}\delta$. Let $h \in H$ be such that $v(h) = \delta$, so that $h \in H \setminus G$ and $H = G \cdot h^{\mathbb{Q}}$. Since $K^{>0} \cap H = G$ and $h \notin G$, it follows that $h \in L \setminus K$, and since K is a real closed field, h is transcendental over K; similarly, $x^\delta \in \boldsymbol{l}((\Delta))$ is transcendental over $K' := \varepsilon_K(K) \subseteq \boldsymbol{k}((\Gamma))$, and thus there exists a unique field isomorphism $\widetilde{\varepsilon}_K : K(h) \longrightarrow K'(x^\delta)$ extending ε_K and mapping h to x^δ. Note that $K'(x^\delta) \subseteq \boldsymbol{k}((\Gamma))(x^\delta) \subseteq \boldsymbol{k}((\Gamma \oplus \mathbb{Z}\delta))$, therefore $\widetilde{\varepsilon}_K$ is the unique field embedding $K(h) \hookrightarrow \boldsymbol{k}((\Gamma \oplus \mathbb{Z}\delta))$ extending ε_K with $\widetilde{\varepsilon}_K(h) = x^\delta$.

Since Γ is divisible and Δ is torsion-free, $n\delta \notin \Gamma$ for all $0 \neq n \in \mathbb{Z}$, and thus given $a := \sum_{i=0}^n a_i h^i \in K[h]$ with $a_i \neq 0$ for all $i \in [n]$, it follows that $v(a_i h^i) \neq v(a_j h^j)$ for all $i, j \in [n]$ with $i \neq j$, therefore

$$
\begin{aligned}
v(a) = v\left(\sum_{i=0}^n a_i h^i\right) &= \min_{0 \le i \le n}\{v(a_i h^i)\} = \min_{0 \le i \le n}\{\nu(\varepsilon(a_i)) + i\delta\} \\
&= \nu\left(\sum_{i=0}^n \varepsilon(a_i)(x^\delta)^i\right) = \nu(\widetilde{\varepsilon}_K(a)),
\end{aligned}
$$

and thus $\widetilde{\varepsilon}_K : K(h) \hookrightarrow \boldsymbol{k}((\Gamma \oplus \mathbb{Z}\delta))$ is an embedding of valued fields; moreover, the value group of $K(h)$ is $\Gamma \oplus \mathbb{Z}\delta$ and its residue field is $\boldsymbol{k}$ ([EP05, Corollary 2.2.3]), therefore $G \cdot h^{\mathbb{Z}} \subseteq K(h)^{>0}$ is a monomial group of $K(h)$ such that $\widetilde{\varepsilon}_K(h') = x^{v(h')}$ for all $h' \in G \cdot h^{\mathbb{Z}}$. By Lemma A.3, $\widetilde{\varepsilon}_K$ is an embedding of ordered valued fields, and since $h \in L = K\langle r\rangle \setminus K$, $K\langle h\rangle = K\langle r\rangle = L$ by the exchange property ([PS86, Theorem 4.1]), and thus it follows that $\widetilde{\varepsilon}_K$ can be extended to an embedding of valued fields $\varepsilon_L : L \hookrightarrow \boldsymbol{l}((\Delta))$.

Case 2. $\Gamma = \Delta$ (hence $G = H$) and there exists $t \in \boldsymbol{l} \setminus \boldsymbol{k}$ such that $\boldsymbol{l} = \boldsymbol{k}\langle t\rangle$. Let $b \in \boldsymbol{l}_0$ be such that $\lambda(b) = t$, so that $b \in \boldsymbol{l}_0 \setminus \boldsymbol{k}_0$ and $\boldsymbol{l}_0 = \boldsymbol{k}_0\langle b\rangle$. Since $K \cap \boldsymbol{l}_0 = \boldsymbol{k}_0$ and $b \notin \boldsymbol{k}_0$, it follows that $b \in L \setminus K$, and since both K and $\boldsymbol{k}_0$ are real closed fields, b is transcendental over both K and $\boldsymbol{k}_0$; similarly, $t \in \boldsymbol{l} \subseteq \boldsymbol{l}((\Delta))$ is transcendental over both $K' := \varepsilon_K(K) \subseteq \boldsymbol{k}((\Gamma))$ and $\boldsymbol{k}$, and thus there exists a unique field isomorphism $\overline{\varepsilon}_K : K(b) \longrightarrow K'(t)$ extending ε_K and mapping b to t. Note that $K'(t) \subseteq \boldsymbol{k}((\Gamma))(t) \subseteq \boldsymbol{k}(t)((\Gamma))$, therefore $\overline{\varepsilon}_K$ is the unique field embedding $K(b) \hookrightarrow \boldsymbol{k}(t)((\Gamma))$ extending ε_K with $\overline{\varepsilon}_K(b) = t$.

Since $v(b) = \nu(\lambda(b)) = \nu(t) = 0$, it follows that both $K \subseteq K(b)$ and $K' \subseteq K'(t)$ are Gauss extensions ([EP05, Corollary 2.2.2]). In particular, given $a := \sum_{i=0}^{n} a_i b^i \in K[b]$ with $a_i \neq 0$ for all $i \in [n]$,

$$v(a) = v\left(\sum_{i=0}^{n} a_i b^i\right) = \min_{0\le i\le n}\{v(a_i)\} = \min_{0\le i\le n}\{\nu(\varepsilon(a_i))\} = \nu\left(\sum_{i=0}^{n} \varepsilon(a_i)t^i\right) = \nu(\overline{\varepsilon}_K(a)),$$

and thus $\overline{\varepsilon}_K : K(b) \hookrightarrow \boldsymbol{k}(t)((\Gamma))$ is an embedding of valued fields; moreover, since $K \subseteq K(b)$ is a Gauss extension, $K(b)$ has value group Γ and residue field $\boldsymbol{k}(t)$, therefore $G \subseteq K^{>0} \subseteq K(b)^{>0}$ is a monomial group of $K(b)$ such that $\overline{\varepsilon}_K(g) = x^{v(g)}$ for all $g \in G$. By Lemma A.3, $\overline{\varepsilon}_K$ is an embedding of ordered valued fields, and arguing as in Case 1 it follows that $\overline{\varepsilon}_K$ can be extended to an embedding of valued fields $\varepsilon_L : L \hookrightarrow \boldsymbol{l}((\Delta))$. □

Remark A.6. Let K be a real closed valued field with value group Γ and residue field $\boldsymbol{k}$. Call a triple $(G, \boldsymbol{k}_0, \varepsilon_K)$ is *admissible* if G is a monomial group of K, $\boldsymbol{k}_0$ is a coefficient field of K, and $\varepsilon_K : K \hookrightarrow \boldsymbol{k}((\Gamma))$ is a valued field embedding such that $\varepsilon_K(a) = \lambda_K(a)$ for all $a \in \boldsymbol{k}_0$ and $\varepsilon_K(g) = x^{v(g)}$ for all $g \in G$. Then the proof of Theorem A.5 shows that if $K \subseteq L$ is an extension of real closed valued fields, then every admissible triple $(G, \boldsymbol{k}_0, \varepsilon_K)$ of K extends (in the obvious sense) to an admissible triple $(H, \boldsymbol{l}_0, \varepsilon_L)$ of L.