

Smooth meets real closed

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Introduction

Classically, algebraic geometry studied solution sets of polynomial equations: $\{\bar{a} \in \mathbb{C}^n \mid p(\bar{a}) = 0\}$ with $p \in \mathbb{C}[x_1, \dots, x_n]$. *Algebraic subsets of $\mathbb{C}^n$* are obtained by taking finite applications of union and intersection of such solution sets.

Geometry was algebraised via $\mathbf{AlgSet}_{\mathbb{C}} \cong \mathbf{f.g.Red} \mathbb{C}\text{-}\mathbf{Alg}^{\text{op}}$. This was abstracted to $\mathbf{AffSch} \cong \mathbf{CRing}^{\text{op}}$ using the *Zariski spectrum* $\text{Spec}(A) = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ a prime ideal}\}$ of a ring A together with a structure sheaf $\mathcal{O}_{\text{Spec}(A)}$ of rings on it.

Real algebraic geometry deals with the study of solution sets of polynomial inequalities: $\{\bar{a} \in \mathbb{R}^n \mid p(\bar{a}) \geq 0\}$ with $p \in \mathbb{R}[x_1, \dots, x_n]$. *Semi-algebraic subsets of $\mathbb{R}^n$* are obtained by taking finite applications of union, intersection and complementation of such solution sets.

In the abstract algebraisation of semi-algebraic geometry, $\text{Spec}(A)$ is replaced by the *real spectrum* $\text{Sper}(A) = \{\alpha \subseteq A \mid \alpha \text{ a prime cone}\}$ of a ring A , and $\mathcal{O}_{\text{Spec}(A)}$ is replaced by a sheaf ρ_A of *real closed rings* on $\text{Sper}(A)$. This yields $\mathbf{AffRCSp} \cong \mathbf{RCR}^{\text{op}}$.

$\mathcal{C}^\infty$ -rings were first introduced to construct topos-theoretic models of synthetic differential geometry via $\mathbf{Man} \hookrightarrow \mathcal{C}^\infty\text{-}\mathbf{Ring}_{\text{f.p.}}^{\text{op}} \hookrightarrow \text{Sh}(\mathcal{C}^\infty\text{-}\mathbf{Ring}_{\text{f.p.}}^{\text{op}})$. Then $\mathcal{C}^\infty$ -algebraic geometry was developed.

Summary: real closed rings and $\mathcal{C}^\infty$ -rings arise as rings attached to geometries: semi-algebraic and differential, respectively.

Real Closed Rings: definition and examples

Definition ([Sch86, PS02])

A ring A is a *real closed ring* if it satisfies the following conditions:

1. A is a reduced ring.
2. The set $\{a^2 \mid a \in A\}$ is the positive cone of a partial order on A such that $(A, \leq)$ is an f -ring.
3. For all $a, b \in A$, if $0 \leq a \leq b$, then there exists $c \in A$ with $a^2 = bc$.
4. For all $\mathfrak{p} \in \text{Spec}(A)$, the field $\text{qf}(A/\mathfrak{p})$ is real closed and $A/\mathfrak{p}$ is integrally closed.

Examples

1. Real closed fields and their convex subrings, that is, *real closed valuation rings* [CD86, CD83, Sch09].
2. Rings $C_{\text{s.a.}}(S)$ of continuous semi-algebraic functions $S \longrightarrow R$ on a semi-algebraic subset $S \subseteq \mathbb{R}^n$ over a real closed field R [Sch89, Chapter III, Section I].
3. Rings $C(X)$ of continuous real-valued functions on a topological space X [Sch97].
4. Rings of global sections of Hausdorff sheaves of real closed valuation rings over Boolean spaces, that is, *Boolean products of real closed valuation rings* [Mac74, BW79, Gui01, Gui25]. See [Sch90, Satz 17] for a generalization.

Real Closed Rings: some algebraic properties

Theorem ([Sch89, Chapter I] and [SM99, Section 12])

Let A be a real closed ring.

1. If $I \subseteq A$ is an ideal, then A/I is real closed if and only if I is a radical ideal.
2. If $S \subseteq A$ is a multiplicative subset, the localization $S^{-1}A$ is real closed.
3. If A is a domain, then A is local.
4. If A is local, then A is Henselian.
5. The support map $\text{supp}_A : \text{Sper}(A) \longrightarrow \text{Spec}(A)$ is a homeomorphism.

Theorem ([Sch87, Chapters II - III], [Sch89, Chapter I] and [SM99, Section 12])

The category **RCR** of real closed rings and ring homomorphisms is complete and cocomplete. The inclusion $\mathbf{RCR} \subseteq \mathbf{CRing}$ has a left adjoint $\rho : \mathbf{CRing} \longrightarrow \mathbf{RCR}$, called the **real closure reflector**. Given $A \in \mathbf{CRing}$, the real closed ring $\rho(A)$ is the **real closure of A** , and A is real closed if and only if the universal morphism $\rho_A : A \longrightarrow \rho(A)$ is an isomorphism.

Examples

$$\rho(\mathbb{C}) = \{0\}, \quad \rho(\mathbb{Q}) = \mathbb{R}_{\text{alg}}, \quad \rho(\mathbb{Q}(\sqrt{2})) = \mathbb{R}_{\text{alg}} \times \mathbb{R}_{\text{alg}}, \quad \rho(\mathbb{R}[x_1, \dots, x_n]) = C_{\text{s.a.}}(\mathbb{R}^n).$$

Real Closed Rings: the real closed structure

Set $\mathbb{C}_{\mathbb{Z}\text{-csa}} := \{f : \mathbb{R}^n \rightarrow \mathbb{R} \mid n \in \mathbb{N}_0, f \text{ is continuous and } \mathbb{Z}\text{-semi-algebraic}\}$. If A is any set define $\mathcal{O}_A := \bigcup_{n \in \mathbb{N}_0} A^{A^n}$.

Theorem ([SM99, Section 12] and [Tre07, Section 2])

A ring A is real closed if and only if there exists an assignment $(-)_A : \mathbb{C}_{\mathbb{Z}\text{-csa}} \rightarrow \mathcal{O}_A$ such that:

1. If f is n -ary, then f_A is n -ary.
2. If $\pi_i^n : \mathbb{R}^n \rightarrow \mathbb{R}$ is the projection onto the i -th coordinate, then $(\pi_i^n)_A : A^n \rightarrow A$ is the projection onto the i -th coordinate.
3. For all $m, n \in \mathbb{N}$, if $f \in \mathbb{C}_{\mathbb{Z}\text{-csa}}$ is m -ary and $g_1, \dots, g_m \in \mathbb{C}_{\mathbb{Z}\text{-csa}}$ are n -ary, then

$$(f \circ (g_1, \dots, g_m))_A = f_A \circ ((g_1)_A, \dots, (g_m)_A).$$

4. $(+_R)_A = +_A$, $(-_R)_A = -_A$, $(\cdot_R)_A = \cdot_A$, $(0_R)_A = 0_A$ and $(1_R)_A = 1_A$.

Moreover:

1. Suppose that A is real closed. Then there is a unique assignment $(-)_A : \mathbb{C}_{\mathbb{Z}\text{-csa}} \rightarrow \mathcal{O}_A$ satisfying 1. - 4. above, and each f_A is definable by an existential Horn formula in the language of rings. The set $\{f_A \mid f \in \mathbb{C}_{\mathbb{Z}\text{-csa}}\}$ is the real closed structure of A .
2. If $F : A \rightarrow B$ is a ring homomorphism between real closed rings, $f \in \mathbb{C}_{\mathbb{Z}\text{-csa}}$ is n -ary and $a_1, \dots, a_n \in A$, then $F(f_A(a_1, \dots, a_n)) = f_B(F(a_1), \dots, F(a_n))$.

$\mathcal{C}^\infty$ -Rings: definition

Let $\mathbf{C}^\infty$ be the category whose objects are $\mathbb{R}^n$ ($n \in \mathbb{N}_0$) together with all smooth maps between them. Let also $\mathcal{C}^\infty := \{f : \mathbb{R}^n \rightarrow \mathbb{R} \mid n \in \mathbb{N}_0, f \text{ is smooth}\}$.

Definition ([Law79, 25-26], [Dub79, Section 1] and [RW78, Example 3])

Then a $\mathcal{C}^\infty$ -ring is a functor $\Phi : \mathbf{C}^\infty \rightarrow \mathbf{Set}$ preserving finite products. Equivalently, $\mathcal{C}^\infty$ -ring is a pair (A, Φ) , where A is a non-empty set and $\Phi : \mathcal{C}^\infty \rightarrow \mathcal{O}_A$ is a function such that:

1. If f is n -ary, then $\Phi(f)$ is n -ary.
2. If $\pi_i^n : \mathbb{R}^n \rightarrow \mathbb{R}$ is the projection onto the i -th coordinate, then $\Phi(\pi_i^n) : A^n \rightarrow A$ is the projection onto the i -th coordinate.
3. For all $m, n \in \mathbb{N}$, if $f \in \mathcal{C}^\infty$ is m -ary and $g_1, \dots, g_m \in \mathcal{C}^\infty$ are n -ary, then

$$\Phi(f \circ (g_1, \dots, g_m)) = \Phi(f) \circ (\Phi(g_1), \dots, \Phi(g_m)).$$

If (A, Φ) is a $\mathcal{C}^\infty$ -ring, then the set $\{\Phi(f) \mid f \in \mathcal{C}^\infty\}$ is the $\mathcal{C}^\infty$ -structure of (A, Φ) . If (B, Ψ) is a $\mathcal{C}^\infty$ -ring, a $\mathcal{C}^\infty$ -morphism $F : (A, \Phi) \rightarrow (B, \Psi)$ is a function $F : A \rightarrow B$ such that for every n -ary $f \in \mathcal{C}^\infty$ ($n \in \mathbb{N}_0$) and all $a_1, \dots, a_n \in A$ the equality

$$F(\Phi(f)(a_1, \dots, a_n)) = \Psi(f)(F(a_1), \dots, F(a_n))$$

holds. Write $\mathcal{C}^\infty$ -Ring for the resulting category.

C^∞ -Rings: examples

If (A, Φ) is a C^∞ -ring, then

$$U(A, \Phi) := (A, \Phi(+_{\mathbb{R}}), \Phi(-_{\mathbb{R}}), \Phi(\cdot_{\mathbb{R}}), \Phi(0), \Phi(1))$$

is an $\mathbb{R}$ -algebra. In particular, say that *an $\mathbb{R}$ -algebra A is a C^∞ -ring* if there exists a function $\Phi : C^\infty \rightarrow \mathcal{O}_A$ satisfying 1. - 3. above and such that $A = U(A, \Phi)$ as $\mathbb{R}$ -algebras.

Examples

1. $C(X)$ and rings $C^\infty(M)$ of smooth real-valued functions on a manifold M .
2. *Formal algebras* [MR91, pp. 18–19], that is, factor rings modulo arbitrary ideals of rings $\mathbb{R}[[x_1, \dots, x_n]]$ of formal power series in $n \in \mathbb{N}$ variables over $\mathbb{R}$.
3. *Weil algebras* [MR91, Definition 3.13 and Theorem 3.17], that is, local $\mathbb{R}$ -algebras W that are finite-dimensional $\mathbb{R}$ -vector spaces such that $W = \mathbb{R} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of W . A concrete example of a Weil algebra is the ring of dual numbers $\mathbb{R}[\varepsilon] := \mathbb{R}[x]/(x^2)$.
4. *Differentiable algebras* in the sense of Navarro González and Sancho de Salas [NGSdS03, Section 2.4], that is, rings isomorphic to a quotient $C^\infty(\mathbb{R}^n)/\mathfrak{a}$ for some $n \in \mathbb{N}$ and some ideal $\mathfrak{a} \subseteq C^\infty(\mathbb{R}^n)$ which is closed in the Fréchet topology of $C^\infty(\mathbb{R}^n)$.

$\mathcal{C}^\infty$ -Rings: some algebraic properties

From [MR86], [MR91, Chapter 1], [CBM20] and [CBTM25, Chapters 4 and 5]:

Theorem

The category $\mathcal{C}^\infty$ -**Ring** is complete and cocomplete and the forgetful functor $\mathcal{C}^\infty$ -**Ring** $\longrightarrow$ **CRing** has a left adjoint. $\mathcal{C}^\infty(\mathbb{R}^n)$ is the free $\mathcal{C}^\infty$ -ring on $n \in \mathbb{N}$ generators, and if E is an infinite set, then the free $\mathcal{C}^\infty$ -ring on E is $\mathcal{C}^\infty(\mathbb{R}^E) := \operatorname{colim}_{F \subseteq_{\text{fin}} E} \mathcal{C}^\infty(\mathbb{R}^F)$.

Theorem

Let A be a $\mathcal{C}^\infty$ -ring.

1. If $I \subseteq A$ is an ideal, then there exists a unique $\mathcal{C}^\infty$ -structure on A/I such that the canonical map $A \longrightarrow A/I$ is a $\mathcal{C}^\infty$ -morphism.
2. If $S \subseteq A$ is any subset, then the localization of A at S in $\mathcal{C}^\infty$ -**Ring** exists. This means that there exists a $\mathcal{C}^\infty$ -ring $A\{S^{-1}\}$ and a $\mathcal{C}^\infty$ -morphism $\eta_S : A \longrightarrow A\{S^{-1}\}$ such that:
 - (a) $\eta_S(S) \subseteq A\{S^{-1}\}^\times$ and
 - (b) For every $\mathcal{C}^\infty$ -morphism $f : A \longrightarrow B$ such that $f(S) \subseteq B^\times$ there exists a unique $\mathcal{C}^\infty$ -morphism $\bar{f} : A\{S^{-1}\} \longrightarrow B$ such that $f = \bar{f} \circ \eta_S$.
3. $\mathcal{C}^\infty$ -fields are real closed fields.
4. $\mathcal{C}^\infty$ -domains are local rings.
5. Local $\mathcal{C}^\infty$ -rings are Henselian.

A Clone-Theoretic Framework: $\mathcal{C}$ -rings

Both $\mathcal{C}_{\mathbb{Z}\text{-csa}}$ and $\mathcal{C}^\infty$ are examples of *(concrete) clones (with nullary operations) on $\mathbb{R}$* [Beh14, Definition 2.1].

By [Tre07, Section 2], a real closed ring is a non-empty set A together with a *clone homomorphism* $\mathcal{C}_{\mathbb{Z}\text{-csa}} \longrightarrow \mathcal{O}_A$. By definition, a $\mathcal{C}^\infty$ -ring is a non-empty set A together with a clone homomorphism $\mathcal{C}^\infty \longrightarrow \mathcal{O}_A$.

If $\mathcal{C}$ is a clone on $\mathbb{R}$ with $\mathcal{C}_{\mathbb{Z}\text{-pol}} \subseteq \mathcal{C}$, a *$\mathcal{C}$ -ring* (A, Φ) is a non-empty set A together with a clone homomorphism $\Phi : \mathcal{C} \longrightarrow \mathcal{O}_A$. The set $\{\Phi(f) \mid f \in \mathcal{C}\}$ is the *$\mathcal{C}$ -structure of the $\mathcal{C}$ -ring (A, Φ)* . The class of $\mathcal{C}$ -rings is a variety in the signature $\{f \mid f \in \mathcal{C}\}$, cf. [KPS14, Section 5].

Examples

1. $\mathcal{C}_{\mathbb{Z}\text{-sa}} := \{f \in \mathcal{O}_{\mathbb{R}} \mid f \text{ is } \mathbb{Z}\text{-semi-algebraic}\}$. The underlying rings of $\mathcal{C}_{\mathbb{Z}\text{-sa}}$ -rings are exactly *semialgebraic function rings* in the sense of Schwartz–Madden (see [SM99, Chapter 7] and [Tre07, Section 2]), equivalently, *von Neumann regular real closed rings* [SM99, Example 12.15].
2. $\mathcal{C}_{\text{cont}} := \{f \in \mathcal{O}_{\mathbb{R}} \mid f \text{ is continuous}\}$. $\mathcal{C}_{\text{cont}}$ -rings are exactly *super real closed rings* in the sense of Tressl [Tre07].
3. $\mathcal{C}_{\mathbb{Z}\text{-Nash}} := \{f \in \mathcal{O}_{\mathbb{R}} \mid f \text{ is } C^\infty \text{ and } \mathbb{Z}\text{-semi-algebraic}\}$. See [BCR98, Section 8] and [Yam16, Yam17].

Smooth Commutative Algebra: $\mathbb{C}^\infty$ -radicals and smooth spectra

Definition ([MR86, p. 329])

Let A be a $\mathbb{C}^\infty$ -ring and $I \subseteq A$ be an ideal. The $\mathbb{C}^\infty$ -radical of I is

$$\sqrt[\infty]{I} := \{a \in A \mid (A/I)\{(a/I)^{-1}\} \cong 0\},$$

where $(A/I)\{(a/I)^{-1}\}$ is the localization of the $\mathbb{C}^\infty$ -ring A/I at a/I in $\mathbb{C}^\infty$ -Ring. An ideal $I \subseteq A$ is a $\mathbb{C}^\infty$ -radical ideal if $I = \sqrt[\infty]{I}$.

$\sqrt[\infty]{I}$ is an ideal for every ideal $I \subseteq A$ of every $\mathbb{C}^\infty$ -ring A , but this is non-trivial, see [CBM20, Theorem 4.28] or [CBTM25, Theorem 5.46]. There exist prime ideals which are not $\mathbb{C}^\infty$ -radical: $\ker(\mathbb{C}^\infty(\mathbb{R}) \twoheadrightarrow \mathbb{R}[[x]]) \subseteq \mathbb{C}^\infty(\mathbb{R})$ [MvQR87, Example 1.2].

Theorem ([MvQR87, Cor. 1.5], [CB18, Thm. 2.3.22], [CBM23, Thm. 6.22])

Let A be a $\mathbb{C}^\infty$ -ring.

1. The $\text{Spec}^\infty(A) := \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \text{ is } \mathbb{C}^\infty\text{-radical}\}$ is a spectral subspace of $\text{Spec}(A)$ and $\text{Spec}^\infty : \mathbb{C}^\infty\text{-Ring} \longrightarrow \mathbf{Spec}$ is a functor to the category of spectral spaces with spectral maps. The space $\text{Spec}^\infty(A)$ is the smooth Zariski spectrum of A .
2. Define the smooth real spectrum of A to be $\text{Sper}^\infty(A) := \text{supp}_A^{-1}(\text{Spec}^\infty(A))$. Then $\text{Sper}^\infty(A)$ is a spectral space and the smooth support map $\text{supp}_A^\infty := (\text{supp}_A)_{\upharpoonright \text{Sper}^\infty(A)}$ is a spectral bijection $\text{supp}_A^\infty : \text{Sper}^\infty(A) \longrightarrow \text{Spec}^\infty(A)$.

Smooth Commutative Algebra: a new description of smooth localizations

What follows is joint work with J. Cerqueira Berni (UFABC), H. L. Mariano (IME-USP) and M. Tressl (University of Manchester, UK).

Let (A, Φ) be a $\mathcal{C}^\infty$ -ring and $S \subseteq A$ be any subset. Recall that the localization of A at S in **CRing** is

$$S^{-1}A = \left\{ \frac{a}{s_1 \cdot \dots \cdot s_n} \mid a \in A, n \in \mathbb{N}, s_i \in S \cup \{1\} \right\}.$$

From [Koc06, p. 156]: “Unlike in ordinary commutative algebra, we do not in general have an explicit description of the elements of $A\{S^{-1}\}$ (the localization of A at S in $\mathcal{C}^\infty$ -**Ring**)”.

Definition (cf. [Tre06, Definition 5.1] and [Tre07, Section 3])

Define $\Upsilon^\infty := \{t : \mathbb{R} \rightarrow \mathbb{R} \mid t \text{ smooth and } t^{-1}(0) = \{0\}\}$. If A is a $\mathcal{C}^\infty$ -ring and $S \subseteq A$ is a subset, write $\Upsilon^\infty(S)$ for the smallest multiplicatively closed subset $M \subseteq A$ with $S \subseteq M$ such that $\Phi(t)(a) \in M$ for all $t \in \Upsilon^\infty$ and all $a \in M$.

Theorem (cf. [Tre07, Theorem 7.4])

Let A be a $\mathcal{C}^\infty$ -ring and $S \subseteq A$ any subset. Then

$$A\{S^{-1}\} \cong \Upsilon^\infty(S)^{-1}A = \left\{ \frac{a}{\Phi(t)(s_1 \cdot \dots \cdot s_n)} \mid a \in A, t \in \Upsilon^\infty, n \in \mathbb{N}, s_i \in S \cup \{1\} \right\}.$$

Smooth Commutative Algebra: consequences of smooth localization

Corollary (cf. [Tre06, Section 5] and [Tre07, Section 6])

Let (A, Φ) be a $\mathcal{C}^\infty$ -ring and $I \subseteq A$ be an ideal. Then

$$\sqrt[\infty]{I} = \{a \in A \mid \exists t \in \Upsilon^\infty \text{ such that } \Phi(t)(a) \in I\}.$$

An element $a \in A$ in a $\mathcal{C}^\infty$ -ring (A, Φ) is a **smooth nilpotent** [BK18, Definition 2] if there exists $t \in \Upsilon^\infty$ with $\Phi(t)(a) = 0$. The $\mathcal{C}^\infty$ -ring A is **$\mathcal{C}^\infty$ -reduced** [MR86, Definition 2.1] if it has no non-zero smooth nilpotents, i.e., $\sqrt[\infty]{(0)} = (0)$. For example, $x \in \mathbb{R}[[x]]$ is a smooth nilpotent in $\mathbb{R}[[x]]$, so $\mathbb{R}[[x]]$ is not $\mathcal{C}^\infty$ -reduced.

Corollary (cf. [CBM20, Proposition 4.51 (3)])

Let A be a $\mathcal{C}^\infty$ -reduced $\mathcal{C}^\infty$ -domain. Then its ring-theoretic quotient field $\text{qf}(A)$ is a $\mathcal{C}^\infty$ -field, hence also a real closed field.

Theorem

Let A be a $\mathcal{C}^\infty$ -ring. The smooth support map

$$\text{supp}_A^\infty : \text{Sper}^\infty(A) \longrightarrow \text{Spec}^\infty(A)$$

is a homeomorphism.

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