Local real closed SV-rings of finite rank and their model theory

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SV-rings

Throughout this talk all rings are commutative and unital.

Definition 1

A ring A is an *SV-ring* if $A/\mathfrak{p}$ is a valuation ring for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

Theorem 2

A ring A is an SV-ring if any of the following equivalent conditions hold:

1. *$A/\mathfrak{p}$ is a valuation ring for all $\mathfrak{p} \in \mathrm{Spec}^{\min}(A)$.*
2. *$A_{\mathrm{red}} := A/\mathrm{Nil}(A)$ is an SV-ring.*
3. *$\mathrm{Spec}(A)$ is normal and $A_{\mathfrak{m}}$ is an SV-ring for all $\mathfrak{m} \in \mathrm{Spec}^{\max}(A)$.*

Lemma 3

If $f : A \twoheadrightarrow C$ and $g : B \twoheadrightarrow C$ are surjective ring homomorphisms of SV-rings, then $A \times_C B := \{(a, b) \in A \times B \mid f(a) = g(b)\}$ is an SV-ring.

Real closed rings: definition and facts

Definition 4

A ring A is a *real closed ring* if it satisfies the following conditions:

1. A is reduced;
2. the set of squares of A is the set of non-negative elements of a partial order $\leq$ on A and $(A, \leq)$ is an f-ring.
3. for all $a, b \in A$, if $0 \leq a \leq b$, then there exists $c \in A$ such that $bc = a^2$; and
4. $\mathrm{qf}(A/\mathfrak{p})$ is a real closed field and $A/\mathfrak{p}$ is integrally closed for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

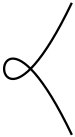
1. The category of real closed rings together with ring homomorphisms is complete and cocomplete.
2. Suppose that A is a real closed ring.
   1. If $I \subseteq A$ is an ideal, then A/I is real closed if and only if I is radical.
   2. If $S \subseteq A$ is a multiplicative subset, then the localization $S^{-1}A$ is real closed.
   3. $\mathrm{supp}_A : \mathrm{Sper}(A) \longrightarrow \mathrm{Spec}(A)$ is a homeomorphism ($\Rightarrow (\mathrm{Spec}(A), \subseteq)$ is a root system).
   4. If $I, J \subseteq A$ are radical ideals, then $I + J$ is a radical ideal. In particular, if $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$ and $1 \notin \mathfrak{p} + \mathfrak{q}$, then $\mathfrak{p} + \mathfrak{q} \in \mathrm{Spec}(A)$.
   5. The poset of radical ideals of A is a distributive lattice with join and meet operations given by sum and intersection of ideals, respectively.
   6. If B is real closed and $f : A \longrightarrow B$ is a ring homomorphism, then f is an f-ring morphism.

Real closed rings: a key geometric example

Let $a \in R$. The *ring of germs of* $C_{\text{s.a}}(R)$ *at* a^+ is $C_{\text{s.a.}}(R)/\mathfrak{p}_{a^+}$, where

$$\mathfrak{p}_{a^+} := \{f \in C_{\text{s.a.}}(R) \mid \exists \varepsilon > 0 \text{ such that } f_{\upharpoonright[a,a+\varepsilon]} = 0\}.$$

The ring $C_{\text{s.a.}}(R)/\mathfrak{p}_{a^-} \cong C_{\text{s.a.}}(R)/\mathfrak{p}_{a^+} =: V$ is a *real closed valuation ring* with residue field R, and the *ring of germs of* $C_{\text{s.a}}(R)$ *at* a is $C_{\text{s.a.}}(R)_{\mathfrak{m}_a} \cong V \times_R V$. Let X be an semi-algebraic curve, e.g. the curve in R^2 given by $y^2 = x^3 + x^2$



Then for all $a \in X$ and for all half-branches β of X at a, the *ring of germs of* $C_{\text{s.a}}(X)$ *at* a^β is $C_{\text{s.a}}(X)/\mathfrak{p}_{a^\beta} \cong V$; in particular, for each $a \in X$, the ring of germs of $C_{\text{s.a}}(X)$ at a is

$$C_{\text{s.a}}(X)_{\mathfrak{m}_a} \cong \underbrace{V \times_R V \cdots \times_R V}_{n_a},$$

where n_a is the number of half-branches of X at a.

The rank of a ring: part 1

Definition 5

Let A be a ring and ∞ be a symbol such that $n < \infty$ for all $n \in \mathbb{N}$.

1. For $\mathfrak{p} \in \mathrm{Spec}(A)$, define $\mathrm{rk}(A, \mathfrak{p}) \in \mathbb{N}$ to be the number of minimal prime ideals $\mathfrak{q}$ of A such that $\mathfrak{q} \subseteq \mathfrak{p}$ if this number is finite, and $\mathrm{rk}(A, \mathfrak{p}) = \infty$ otherwise.
2. The *rank of A* is $\mathrm{rk}(A) := \sup\{\mathrm{rk}(A, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\} \in \mathbb{N} \cup \{\infty\}$.

The ring A is *of finite rank* if $\mathrm{rk}(A) \neq \infty$.

Lemma 6

Let A be a ring. The following are equivalent:

1. *A is a reduced local SV-ring of rank 1.*
2. *A is a valuation ring.*

Proof.

If item (i) holds, then A being local of rank 1 implies that A has exactly one minimal prime ideal $\mathfrak{q}$, therefore $\mathrm{Nil}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p} = \mathfrak{q}$ is a prime ideal of A; since A is reduced, $\mathfrak{q} = (0)$, and since A is an SV-ring, $A/\mathfrak{q} = A$ is a valuation ring, as required. □

The rank of a ring: part 2

Local domains are exactly reduced local rings of rank 1, therefore a reduced local ring of rank 2 has zero divisors. The next lemma clarifies the relationship between the rank of a reduced local ring and its zero divisors.

Definition 7

Let A be a ring. Two elements $a, b \in A$ are *orthogonal* if $ab = 0$.

Lemma 8

Let A be a reduced local ring and set $\mathrm{Spec}^{\mathrm{min}}(A) = \{\mathfrak{p}_i \mid i \in I\}$.

1. $\mathrm{rk}(A) = \sup\{m \in \mathbb{N} \mid \exists a_1, \ldots, a_m \in A \text{ non-zero and pairwise orthogonal}\}$.
2. *If* $\mathrm{rk}(A) = |I| = n \in \mathbb{N}^{\geq 2}$, *then for all* $a_1, \ldots, a_n \in A$ *non-zero and pairwise orthogonal there exists a bijection* $\sigma : [n] \longrightarrow [n]$ *such that* $\mathfrak{p}_i = \mathrm{Ann}(a_{\sigma(i)})$ *for all* $i \in [n]$, *where* $\mathrm{Ann}(a) := \{b \in A \mid ab = 0\}$ *for* $a \in A$.

Local real closed (SV-)rings of finite rank

Thought the rest of the talk fix $n \in \mathbb{N}$ with $n \geq 2$.

Theorem 9

Let A be a local real closed (SV-)ring of rank n with $\mathrm{Spec}^{\min}(A) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. *The canonical embedding $A \hookrightarrow \prod_{i=1}^n A/\mathfrak{p}_i$ given by $a \mapsto (a/\mathfrak{p}_1, \dots, a/\mathfrak{p}_n)$ corestricts to an isomorphism*

$$A \cong \left(\left(\left(\frac{A}{\mathfrak{p}_1} \times_{\frac{A}{\mathfrak{q}_2}} \frac{A}{\mathfrak{p}_2} \right) \times_{\frac{A}{\mathfrak{q}_3}} \frac{A}{\mathfrak{p}_3} \right) \cdots \times_{\frac{A}{\mathfrak{q}_{n-1}}} \frac{A}{\mathfrak{p}_{n-1}} \right) \times_{\frac{A}{\mathfrak{q}_n}} \frac{A}{\mathfrak{p}_n},$$

where $\mathfrak{q}_j := (\bigcap_{i=1}^{j-1} \mathfrak{p}_i) + \mathfrak{p}_j$ for all $j \in \{2, \dots, n\}$.

Let A be as in Theorem 9 and $j \in \{2, \dots, n\}$. Since A is a local real closed ring:

$$\mathfrak{q}_j = \left(\bigcap_{i=1}^{j-1} \mathfrak{p}_i \right) + \mathfrak{p}_j = \bigcap_{i=1}^{j-1} (\mathfrak{p}_i + \mathfrak{p}_j) = \mathfrak{p}_{i_0} + \mathfrak{p}_j \in \mathrm{Spec}(A),$$

where $i_0 \in \{1, \dots, n-1\}$ is such that $\mathfrak{p}_{i_0} + \mathfrak{p}_j \subseteq \mathfrak{p}_i + \mathfrak{p}_j$ for all $i \in \{1, \dots, n-1\}$.

Branching ideals

Definition 10

Let A be a ring. A prime ideal $\mathfrak{q} \subseteq A$ is a *branching ideal* if there exist distinct $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}(A)$ such that $\mathfrak{p}_1, \mathfrak{p}_2 \subsetneq \mathfrak{q}$ and $\mathfrak{q} = \mathfrak{p}_1 + \mathfrak{p}_2$.

Lemma 11

Let A be a real closed ring. A prime ideal $\mathfrak{q} \subseteq A$ is a branching ideal if there exist distinct $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}^{\mathrm{min}}(A)$ such that $\mathfrak{p}_1, \mathfrak{p}_2 \subsetneq \mathfrak{q}$ and $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2$.

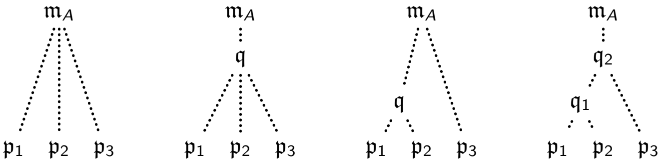
Proposition 12

Let A be a local real closed ring of rank n. The following are equivalent:

1. *$\mathfrak{m}_A$ is a branching ideal.*
2. *Every non-unit of A is a sum of two zero divisors.*
3. *There exist $r, s \in [n]$ and local real closed rings A_1 and A_2 with isomorphic residue field K such that $r + s = n$, $\mathrm{rk}(A_1) = r$, $\mathrm{rk}(A_2) = s$, and $A \cong A_1 \times_K A_2$.*

Rings of type $(n, 1)$ and of type $(n, 2)$

The next diagram summarizes all possible configurations of minimal prime ideals and branching ideals in a local real closed ring of rank 3:



Definition 13

Let A be a ring.

1. A is of *type* $(n, 1)$ if A is a local real closed SV-ring of rank n with exactly one branching ideal $\mathfrak{b}_A$, which moreover is maximal.
2. A is of *type* $(n, 2)$ if A is a local real closed SV-ring of rank n with exactly one branching ideal $\mathfrak{b}_A$, which moreover is not maximal.

Rings of type $(n, 1)$ (of type $(n, 2)$) are n-fold fibre products $V_1 \times_D V_2 \times \cdots \times_D V_n$ of non-trivial real closed valuation rings V_i along surjective maps $V_i \twoheadrightarrow D$ onto a field (domain) by Theorem 9.

The theories T_n, $T_{n,1}$, and $T_{n,2}$: part 1

Throughout the rest of the talk let $\mathscr{L} := \{+, -, \cdot, 0, 1\}$ be the language of rings.

Let $\varphi_{\mathrm{rk}=n}$ be the $\mathscr{L}$-sentence expressing:

"there exists n non-zero pairwise orthogonal elements $a_1, \ldots, a_n \in A$ such that if $b \in A$ is a non-zero element distinct from all the a_i, then b is not orthogonal to any a_i".

Let A be a reduced local ring. Then $A \models \varphi_{\mathrm{rk}=n}$ if and only if A has rank n by Lemma 8. Suppose now that $A \models \varphi_{\mathrm{rk}=n}$. Then $\mathrm{Spec}^{\min}(A) = \{\mathrm{Ann}(a_i) \mid i \in [n]\}$ for any non-zero pairwise orthogonal elements $a_1, \ldots, a_n \in A$, therefore A is an SV-ring if for all $b, c \in A$ and all $i \in [n]$, $b/\mathrm{Ann}(a_i)$ divides $c/\mathrm{Ann}(a_i)$ or $c/\mathrm{Ann}(a_i)$ divides $b/\mathrm{Ann}(a_i)$ by Theorem 2. That is,

$$A \models \exists x((bx - c)a_i = 0) \text{ or } A \models \exists x((cx - b)a_i = 0).$$

Let $\varphi_{\mathrm{SV},n}$ be the $\mathscr{L}$-sentence expressing:

"for all non-zero pairwise orthogonal elements $a_1, \ldots, a_n \in A$, for all $b, c \in A$, and for all $i \in [n]$, either ba_i divides ca_i or ca_i divides ba_i".

Definition 14

Let T_n be the theory of local real closed rings together with $\varphi_{\mathrm{rk}=n}$ and $\varphi_{\mathrm{SV},n}$.

The theories T_n, $T_{n,1}$, and $T_{n,2}$: part 2

Let $A \models T_n$. Then $\mathrm{Spec}^{\min}(A) = \{\mathrm{Ann}(a_i) \mid i \in [n]\}$ by Lemma 8, and thus every branching ideal of A is of the form $\mathrm{Ann}(a_i) + \mathrm{Ann}(a_j)$ for some $i, j \in [n]$ with $i \neq j$ by Lemma 11. Let $\varphi_{\mathrm{br},n}$ to be the $\mathscr{L}$-sentence expressing:

"$\mathrm{Ann}(a_i) + \mathrm{Ann}(a_j) = \mathrm{Ann}(a_k) + \mathrm{Ann}(a_\ell)$ for all pairwise orthogonal non-zero elements $a_1, \dots, a_n \in A$ and for all $i, j, k, \ell \in [n]$ such that $i \neq j$ and $k \neq \ell$."

Therefore $A \models \varphi_{\mathrm{br},n}$ if and only if A has exactly one branching ideal.

Definition 15

Let $T_{n,1}$ be the theory $T_n \cup \{\varphi_{\mathrm{br},n}\}$ together with the $\mathscr{L}$-sentence expressing: "every unit is a sum of two zero divisors". Let $T_{n,2}$ be the theory $T_n \cup \{\varphi_{\mathrm{br},n}\}$ together with the $\mathscr{L}$-sentence expressing: "there exists a unit which is not a sum of two zero divisors".

Models of $T_{n,j}$ are exactly rings of type (n, j) for $j \in \{1, 2\}$ by Proposition 12. Note that if $A \models T_{n,2}$, then the branching ideal $\mathfrak{b}_A$ is definable (without parameters) by the formula $\mathfrak{b}(x)$ expressing

"there exist pairwise orthogonal non-zero elements $a_1, \dots, a_n \in A$ such that $x \in \mathrm{Ann}(a_1) + \mathrm{Ann}(a_2)$".

Model completeness for $T_{n,1}$

Theorem 16

$T_{n,1}$ is model complete, that is, if $A, B \models T_{n,1}$ and $A \subseteq B$ as $\mathscr{L}$-structures, then A is existentially closed in B (if $\varphi(x)$ is a finite Boolean combination of formulas of the form $f(x) = 0$ with $f \in A[x]$, then $B \models \exists x\varphi(x)$ implies $A \models \exists x\varphi(x)$).

Sketch of proof.

Let $A, B \models T_{n,1}$ such that $A \subseteq B$ as $\mathscr{L}$-structures. Set $\mathbf{k} := A/\mathfrak{m}_A$ and $\mathbf{l} := B/\mathfrak{m}_B$. There exists a commutative diagram of local embeddings

$$\begin{array}{ccc} A' & \xrightarrow{\ \eta\ } & B' \\ \uparrow{\scriptstyle \varepsilon_A} & & \uparrow{\scriptstyle \varepsilon_B} \\ A & \xrightarrow{\ \subseteq\ } & B \end{array}$$

where $A' := \mathbf{k}[[\Gamma]] \times_{\mathbf{k}} \mathbf{k}[[\Gamma]] \ldots \times_{\mathbf{k}} \mathbf{k}[[\Gamma]]$ and $B' := \mathbf{l}[[\Delta]] \times_{\mathbf{l}} \mathbf{l}[[\Delta]] \ldots \times_{\mathbf{l}} \mathbf{l}[[\Delta]]$ for some divisible o-groups Γ and Δ, and $\mathbf{k}[[\Gamma]]$ is the valuation ring of the real closed Hahn series field $\mathbf{k}((\Gamma))$. Then A' is existentially closed in B' and A is existentially closed in A'. □

Model completeness for $T_{n,2}$ and consequences of model completeness

Theorem 17

Let $T^*_{n,2}$ be the $(\mathscr{L} \cup \{\mathtt{m}, \mathtt{b}\})$-theory extending $T_{n,2}$ and stating that $\mathtt{m}$ is the maximal ideal and $\mathtt{b}$ is the branching ideal. Then $T^*_{n,2}$ is model complete.

Theorem 18

$T_{n,1}$ is the model companion of the theory of local real closed (SV-) rings of rank n.

Sketch of proof.

Let $A \models T_n$ and $\mathrm{Spec}^{\min}(A) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Then there exists a canonical $\mathscr{L}$-embedding

$$A \hookrightarrow \left(\left(\frac{A}{\mathfrak{p}_1} \times_{\frac{A}{\mathfrak{m}_A}} \frac{A}{\mathfrak{p}_2} \right) \cdots \times_{\frac{A}{\mathfrak{m}_A}} \frac{A}{\mathfrak{p}_{n-1}} \right) \times_{\frac{A}{\mathfrak{m}_A}} \frac{A}{\mathfrak{p}_n} \models T_{n,1}.$$

□

Theorem 19

$T_{n,1}$ and $T_{n,2}$ are complete, hence decidable.

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