Relative Quantifier Elimination for Lattice-Ordered Modules of Continuous Semi-Algebraic Functions on a Curve

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Contents

Some Model Theory of ℓ-Groups: standard structures

Convention. In this talk, all groups are abelian.

Let G be an ℓ-group of functions $X \longrightarrow N$ ($X \neq \varnothing$ a set, N an o-group) and $f \in G$.

- $\{f = 0\} := \{x \in X \mid f(x) = 0\}$ and $\{f \geq 0\} := \{x \in X \mid f(x) \geq 0\}$; note that

$$\{f = 0\} = \{f \geq 0\} \cap \{-f \geq 0\} \text{ and } \{f \geq 0\} = \{f \wedge 0 = 0\}.$$

- $L_{G,X,N} := \{\{f = 0\} \mid f \in G\} = \{\{f \geq 0\} \mid f \in G\}$; note that

$$\{f \geq 0\} \cup \{g \geq 0\} = \{f \vee g \geq 0\} \text{ and } \{f \geq 0\} \cap \{g \geq 0\} = \{f \wedge g \geq 0\}.$$

- $\mathscr{L}^{\mathrm{gr}} := \{+, -, 0\}$, $\mathscr{L}^{\ell\text{-gr}} := \mathscr{L}^{\mathrm{gr}} \cup \{\leq, \wedge, \vee\}$, and $\mathscr{L}^{\mathrm{lat}} := \{\sqsubseteq, \sqcap, \sqcup, \top\}$.

Definition

The *standard structure for* $G \subseteq N^X$ is the 2-sorted structure $(G, P, L_{G,X,N})$, where:

- G is regarded as an $\mathscr{L}^{\ell\text{-gr}}$-structure,
- $L_{G,X,N}$ is regarded as an $\mathscr{L}^{\mathrm{lat}}$-structure, and
- $P : G \longrightarrow L_{G,X,N}$ is the map $f \mapsto \{f \geq 0\}$.

Some Model Theory of ℓ-Groups: the patching condition

Definition

$G \subseteq N^X$ is *closed under patching in* N^X if for all $f, g \in G$ and for all $A, B \in L_{G,X,N}$,

$$f_{\restriction A\cap B} = g_{\restriction A\cap B} \implies \exists h \in G \text{ such that } h_{\restriction A} = f_{\restriction A} \;\&\; h_{\restriction B} = g_{\restriction B};$$

equivalently,

$$(G, P, L_{G,X,N}) \models \forall xy \forall \zeta_1\zeta_2[\zeta_1 \sqcap \zeta_2 \sqsubseteq \{x = y\} \rightarrow \exists z(\zeta_1 \sqsubseteq \{z = x\} \;\&\; \zeta_2 \sqsubseteq \{z = y\})].$$

Example. Let N be a real closed field, $X \subseteq N^n$ be closed and semi-algebraic, and G be the ℓ-group of all continuous semi-algebraic functions $X \longrightarrow N$. Let $f, g \in G$ and $A, B \in L_{G,X,N}$ be such that $f_{\restriction A\cap B} = g_{\restriction A\cap B}$. The map $h_0 : A \cup B \longrightarrow N$ given by

$$h_0(x) := \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous and semi-algebraic. By the semi-algebraic Tietze extension theorem there exists $h \in G$ such that $h_{\restriction A\cup B} = h_0$; such h verifies the patching condition for G.

Some Model Theory of ℓ-Groups: the Shen-Weispfenning theorem

Theorem (Shen, Weispfenning)

Let $\varphi(\overline{x}, \overline{\xi})$ be an $(\mathscr{L}^{\ell\text{-gr}} \cup \mathscr{L}^{\text{lat}} \cup \{P\})$-formula. Then $\varphi(\overline{x}, \overline{\xi})$ is effectively equivalent in every divisible standard structure closed under patching to a formula of the form

$$\exists \zeta_1 \dots \zeta_m \left[\gamma(\overline{\xi}, \overline{\zeta}) \ \& \ \bigwedge_{i=1}^{m} \zeta_i = P(t_i(\overline{x})) \right],$$

where ζ_i are lattice variables, γ is an $\mathscr{L}^{\text{lat}}$-formula, and t_i are $\mathscr{L}^{\text{gr}}$-terms.

Let $(G, P, L_{G,X,N})$ be a standard structure and $f, g \in G$. Then

$$(G, P, L_{G,X,N}) \models f \leq g \leftrightarrow P(g - f) = \top \leftrightarrow \exists \zeta [\zeta = \top \ \& \ \zeta = P(g - f)].$$

General proof idea: reduce the problem to eliminating $\mathscr{L}^{\ell\text{-gr}}$-quantifiers in formulas of simple form (e.g. $\exists z[\xi_1 \sqsubseteq \{z \geq x\} \ \& \ \xi_2 \sqsubseteq \{z \leq y\}]$); then use the patching condition to eliminate such quantifiers (e.g. $\xi_1 \sqcap \xi_2 \sqsubseteq \{x \leq y\}$).

Corollary

Let $G \subseteq N^X$ be divisible and closed under patching. If the lattice $L_{G,X,N}$ is decidable, then so is the standard structure $(G, P, L_{G,X,N})$, as well as the ℓ-group G.

The Set-Up

Fix the following notation:

- R is a real closed field.
- $X \subseteq R^n$ is a semi-algebraic curve.
- $C_{\text{s.a.}}(X)$ is the set of continuous semi-algebraic functions $X \longrightarrow R$.
- $L := L_{C_{\text{s.a.}}(X),X,R}$; i.e., L is the lattice of closed and semi-algebraic subsets of X.

Note that

$$(C_{\text{s.a.}}(X); +, -, 0, \leq, \wedge, \vee) \xrightarrow{P} (L; \subseteq, \cap, \cup, \top)$$

is a divisible standard structure closed under patching.

Consider the structure

$$\mathscr{M}_0 := (C_{\text{s.a.}}(X); +, -, 0, \leq, \wedge, \vee, (\alpha)_{\alpha \in C_{\text{s.a.}}(X)}) \xrightarrow{P} (L; \subseteq, \cap, \cup, \top);$$

following the Shen-Weispfenning proof, one can show that eliminating module quantifiers in formulas modulo $\mathscr{M}_0$ reduces to eliminating module quantifiers in formulas of simple form, e.g.

$$\exists z[\xi_1 \sqsubseteq \{\alpha z \geq x\} \;\&\; \xi_2 \sqsubseteq \{\beta z \leq y\}],$$

where $\alpha, \beta \geq 0$

Eliminating Module Quantifiers: local solutions to systems of inequalities

Lemma

Let $A, B \in L$, $f, g \in C_{\text{s.a.}}(X)$ and $\alpha, \beta \geq 0$. Let $\{a_1, \dots, a_n\}$ be the union of the boundary points of $\{\alpha = 0\}$ and of $\{\beta = 0\}$. The following are equivalent:

(i) *There exists $h \in C_{\text{s.a.}}(X)$ such that*

$$A \subseteq \{\alpha h \geq f\} \text{ and } B \subseteq \{\beta h \leq g\}. \quad (\dagger)$$

(ii) **(a)** $A \cap B \subseteq \{\beta f \leq \alpha g\}$, $A \cap \{\alpha = 0\} \subseteq \{0 \geq f\}$, $B \cap \{\beta = 0\} \subseteq \{0 \leq g\}$, *and*
(b) *there exists $\epsilon > 0$ such that $(\dagger)$ is solvable in $Z_0 := \bigcup_{i=1}^{n} \overline{B}_\epsilon(a_i)$; i.e. there exists $h_0 \in C_{\text{s.a.}}(X)$ such that $Z_0 \cap A \subseteq \{\alpha h_0 \geq f\}$ and $Z_0 \cap B \subseteq \{\beta h_0 \leq g\}$.*

That is, there exists $h \in C_{\text{s.a.}}(X)$ solving $A \subseteq \{\alpha x \geq f\}$ and $B \subseteq \{\beta x \leq g\}$ if and only if:

(a) some inclusions of zero sets hold (expressible in $\mathscr{M}_0$ with parameters), and
(b) there exists $h \in C_{\text{s.a.}}(X)$ solving $A \subseteq \{\alpha x \geq f\}$ and $B \subseteq \{\beta x \leq g\}$ locally around certain finitely many points of X.

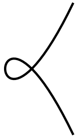
Main idea to eliminate module quantifiers: enrich $\mathscr{M}_0$ with its germwise structure to express item **(b)** in a first-order way.

Rings of Germs of Continuous Semi-Algebraic Functions on a Curve

Let $a \in R$. The *ring of germs of* $C_{\text{s.a}}(R)$ *at* a^+ is $C_{\text{s.a.}}(R)/\mathfrak{p}_{a^+}$, where

$$\mathfrak{p}_{a^+} := \{f \in C_{\text{s.a.}}(R) \mid \exists \epsilon > 0 \text{ such that } f_{\restriction [a,a+\epsilon]} = 0\};$$

note that $C_{\text{s.a.}}(R)/\mathfrak{p}_{a^-} \cong C_{\text{s.a.}}(R)/\mathfrak{p}_{a^+} =: V$ is a *real closed valuation ring* with residue field R, and the *ring of germs of* $C_{\text{s.a}}(R)$ *at* a is $C_{\text{s.a.}}(R)_{\mathfrak{m}_a} \cong V \times_R V$. If X is an arbitrary curve, e.g. the curve in R^2 given by $y^2 = x^3 + x^2$



then for all $a \in X$ and for all half-branches a^κ of X at a, the *ring of germs of* $C_{\text{s.a}}(X)$ *at* a^κ is $C_{\text{s.a}}(X)/\mathfrak{p}_{a^\kappa} \cong V$; in particular, for each $a \in X$, the ring of germs of $C_{\text{s.a}}(X)$ at a is

$$C_{\text{s.a}}(X)_{\mathfrak{m}_a} \cong \underbrace{V \times_R V \cdots \times_R V}_{n_a},$$

where n_a is the number of half-branches of X at a.

The 3-Sorted Structure $\mathscr{M}$

Definition

Let $\mathscr{M}$ be the following 3-sorted structure:

$$(L; \subseteq, \cap, \cup, \top)$$

$$(C_{\text{s.a.}}(X); +, -, 0, \leq, \wedge, \vee, (\alpha)_{\alpha \in C_{\text{s.a.}}(X)}) \xrightarrow{\;P\;} (L; \subseteq, \cap, \cup, \top)$$

$$(C_{\text{s.a.}}(X); +, -, 0, \leq, \wedge, \vee, (\alpha)_{\alpha \in C_{\text{s.a.}}(X)}) \xrightarrow{\{\pi_{a^\kappa} \mid \; a \in X \;\; \& \;\; a^\kappa \text{ a half-branch of } X \text{ at } a\}} (V; +, -, \cdot, 0, 1, \leq)$$

where π_{a^κ} is a unary function symbol $C_{\text{s.a.}}(X) \longrightarrow V$ ($\cong C_{\text{s.a.}}(X)/\mathfrak{p}_{a^\kappa}$) interpreted as the map $f \mapsto f/\mathfrak{p}_{a^\kappa}$ for each $a \in X$ and each half-branch a^κ of X at a.

Eliminating Module Quantifiers in $\mathscr{M}$: an example via local divisibility

Example. Suppose that $X = R$, $A = [-1, 1]$, and $\partial(\{\alpha = 0\}) = \{0\}$. The following are equivalent:

(i) $\mathscr{M} \models \exists x[A \sqsubseteq \{\alpha x = f\}]$.
(ii) (a) $\mathscr{M} \models A \cap \{\alpha = 0\} \subseteq \{f = 0\}$, and
(b) $\mathscr{M} \models \exists X_1, X_2[\pi_{0^-}(\alpha)X_1 = \pi_{0^-}(f)\ \&\ \pi_{0^+}(\alpha)X_2 = \pi_{0^+}(f)\ \&\ X_1 - X_2 \in \mathfrak{m}]$,

where $X_1 - X_2 \in \mathfrak{m}$ is the formula in V stating that $X_1 - X_2$ is not a unit.

Proof. By the Lemma in slide 7, **(i)** is equivalent to **(a)** together with the statement $\exists \epsilon > 0 \exists h_0 \in C_{\text{s.a.}}(X)[\overline{B}_\epsilon(0) \subseteq \{\alpha h_0 = f\}]$; this latter statement says exactly that α divides f locally around 0, i.e. $C_{\text{s.a.}}(X)_{\mathfrak{m}_0} \cong V \times_R V \models \exists X[\alpha_{\mathfrak{m}_0} X = f_{\mathfrak{m}_0}]$, and this is equivalent to **(b)**. □

Statement of the Main Theorem

Theorem (P.P.)

Let $\varphi(\overline{x}, \overline{\xi}, \overline{X})$ be a formula in the language of $\mathscr{M}$. Then $\varphi(\overline{x}, \overline{\xi}, \overline{X})$ is equivalent in $\mathscr{M}$ formula of the form

$$\exists \zeta_1 \dots \zeta_m \exists Z_1 \dots Z_n \left[\gamma(\overline{\xi}, \overline{X}, \overline{\zeta}, \overline{Z}) \mathbin{\&} \bigwedge_{i=1}^{m} \zeta_i = P(s_i(\overline{x})) \mathbin{\&} \bigwedge_{j=1}^{n} Z_j = \pi_j(t_j(\overline{x})) \right], \qquad (\dagger)$$

where:

- ζ_i *are variables from* L,
- Z_j *are variables from* V,
- γ *is a Boolean combination of formulas (with parameters) in* L *and of formulas (with parameters) in* V,
- s_i *and* t_j *are module terms, and*
- *each* π_j *is* π_{a^κ} *for some* $a \in X$ *and some half-branch* a^κ *of* X *at* a.

Moreover, if R is a recursive real closed field, then $\varphi(\overline{x}, \overline{\xi}, \overline{X})$ is effectively equivalent to a formula of the form $(\dagger)$.

A Corollary of the Main Theorem

Corollary

Suppose that R is a recursive real closed field. Then the 3-sorted structure $\mathscr{M}$ is decidable, and thus so is the ℓ-module $C_{\mathrm{s.a.}}(X)$.

Sketch of proof.

By the Main Theorem, it suffices to check that both L and V are decidable. L is decidable because it is a “1-dimensional topological lattice” (Tressl); V is decidable because its theory has a recursive axiomatization (Cherlin-Dickmann). □

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