Maths at University and Infinite Sets

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Who is Ricardo?

- Originally from Spain.
- Currently studying the 4th year of the MMath degree at The University of Manchester.
- Starting a PhD in Mathematical Logic from September 2020, also at The University of Manchester.
- Mathematical interests: Mathematical Logic (Set Theory, Model Theory and Category Theory) and Algebra (Homological Algebra).

What is Mathematics anyway?

Question: In a few words, how would you describe Mathematics?

What is Mathematics anyway?

- Science which studies structure, order and relations.
- Uses as a tools logical reasoning and valid deduction arguments.
- It's main goal is to establish and study the true properties of different mathematical objects, such as numbers, equations and geometrical shapes (among many others).

What is Mathematics anyway?

- Some of the first uses of Mathematics date as far back as 3000 BC.
- Ancient Babylonians and Egyptians started to develop Mathematics in order to solve practical problems such as calculating areas for harvesting, establishing taxes or dividing food in equal amounts for a group of people.
- Nowadays Mathematics has developed so much there are at least 60 different areas, each of which subdivides into more subdisciplines.

Pure Mathematics

- Algebra.
- Geometry.
- Mathematical Logic.

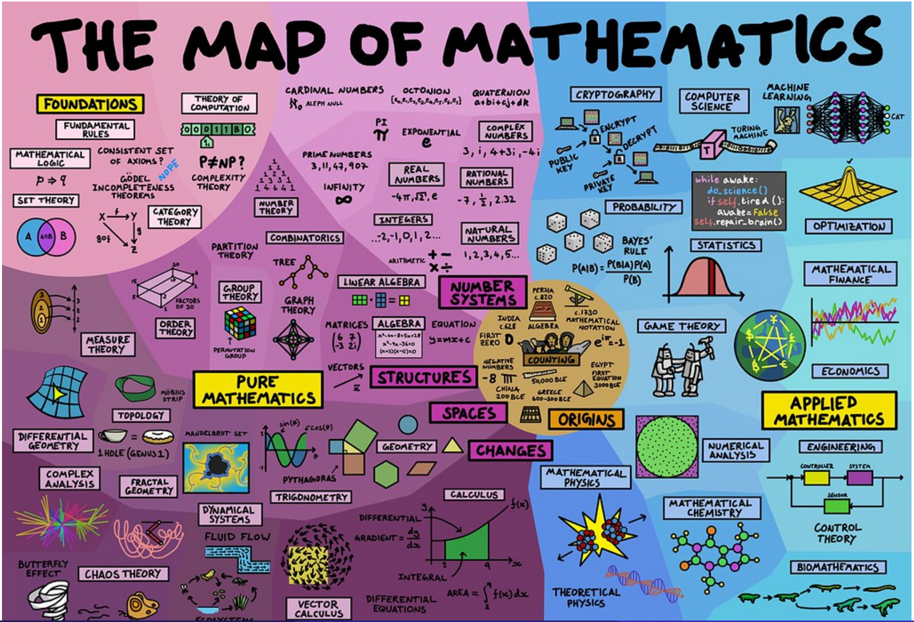
Applied Mathematics

- Mathematical Physics.
- Optimization.
- Mathematical Biology.

Statistics

- Mathematical Finance.
- Machine Learning.
- Experiment Design and Analysis.

What is Mathematics anyway?



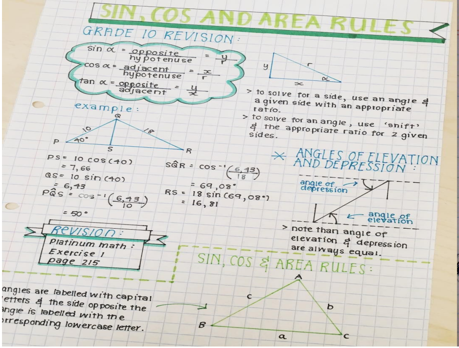
Question: Can you think af an example of high-school Maths which mixes two of the areas of Mathematics shown before?

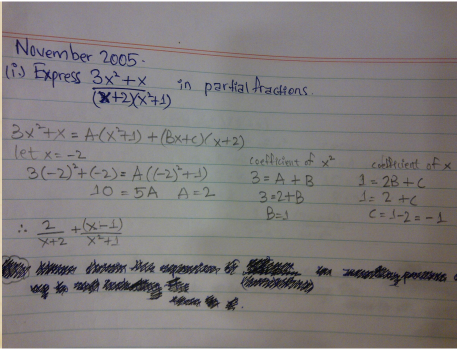
Coordinate Geometry is one example; it's one of the many ways that Geometry (lines, curves, polygons) interacts with Algebra (equations).

The first big question to answer is: *"How does one go from high-school Mathematics to University level Mathematics?"*

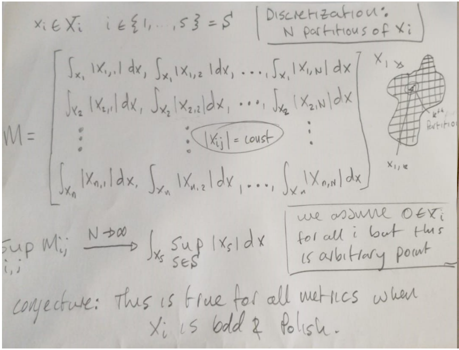
In other words...

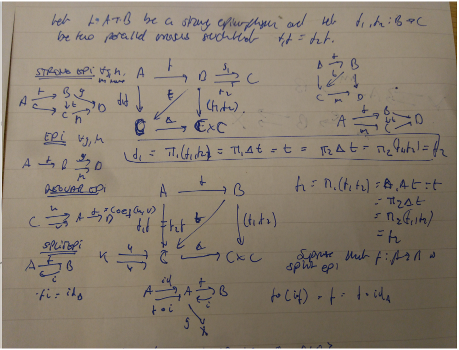
"How does one go from these..."





"... to these?"


Discretization:
N partitions of Xi
we assume 0 ∈ Xi
for all i but this
is arbitrary point
conjecture: This is true for all metrics when
Xi is bdd & Polish.



Mathematics at University

- The first step is learning the language!
- This is the main focus of the first two years of the Mathematics degree at University.
- New mathematical vocabulary and new mathematical objects.
- Learning how to prove results.

1st difference with high-school Maths: at University, *one learns Mathematics through understanding properties and proving results*, relying less on memory or computational accuracy.

Mathematics at University

- One could say that, in a nutshell, studying Mathematics (specially Pure and Applied) is learning how to prove things.
- Mathematics aims to abstract the observed structure and to generalize patterns.
- The abstraction comes in shape of *definitions*.
- The results and generalized patterns come in shape of *Theorems* and other true statements, namely *Lemmas, Corollaries and Propositions*.

2nd difference with high-school Maths: at University, *Mathematics revolves around properly defining general mathematical objects* and *proving results about them*, instead of focusing on concrete examples or numerical calculations.

Two examples on how University level Mathematics "looks like":

Definition 1.5. A *G-subspace* of a *G*-space V is a vector subspace $W \leq V$ such that $gW \leq W$ for all $g \in G$.

W is itself a G-space (easy exercise).

Definition 1.6. If V is a G-space and W is a G-subspace then the *quotient space* V/W is a G-space with the action $g(v + W) = gv + W$.

In terms of matrices we have

$$\rho_V = \left[\begin{array}{c|c} \rho_W & ? \\ \hline 0 & \rho_{V/W} \end{array}\right].$$

Definition 1.7. If V and W are two G-spaces we make their usual *direct sum* $V \oplus W$ into a G-space by $g(v, w) = (gv, gw), \quad g \in G,\ v \in V,\ w \in W$.

$$\rho_{V \oplus W} = \left[\begin{array}{c|c} \rho_V & 0 \\ \hline 0 & \rho_W \end{array}\right].$$

Lemma 1.8. *Let $f: V \to W$ be a homomorphism of G-spaces. Then:*

(1) $\mathrm{im}(f)$ *is a G-subspace of W,*
(2) $\ker(f)$ *is a G-subspace of V.*

Proof. (1) Certainly $\mathrm{im}(f)$ is a vector subspace of W. If $x \in \mathrm{im}(f)$ then $x = f(y)$ for some $y \in V$. For any $g \in G$, we have $gx = gf(y) = f(gy)$, so $gx \in \mathrm{im}(f)$. Thus $g\,\mathrm{im}(f) \leq \mathrm{im}(f)$, as required.

(2) Clearly $\ker(f)$ is a vector subspace of V. If $z \in \ker(f)$ then $f(z) = 0$, so $f(gz) = gf(z) = g0 = 0$. Thus $g\ker(f) \leq \ker(f)$, as required. □

Representation Theory (Year 4)

1.18 Definition. A random variable X is said to be *discrete* if its range R_X is a countable set, i.e. either it is finite $R_X = \{ x_i \mid 1 \leqslant i \leqslant n \}$ or it is denumerable $R_X = \{ x_i \mid i \geqslant 1 \}$.

1.19 Definition. [MATH10141, 4.1.3.] Given a discrete random variable X, we may define the *probability mass function* (pmf) of X to be the function $p_X : \mathbb{R} \to [0,1]$ determined by $p_X(x) = \mathbf{P}(X = x)$.

1.20 Proposition. Given a discrete random variable X with range $R_X = \{x_i\}$, then, for any subset $A \subseteq \mathbb{R}$,

$$\mathbf{P}(X \in A) = \sum_{x_i \in A} p_X(x_i).$$

In particular,

$$F_X(a) = \mathbf{P}(X \leqslant a) = \sum_{x_i \leqslant a} p_X(x_i),$$

$$\mathbf{P}(X < a) = \sum_{x_i < a} p_X(x_i),$$

for all $a \in \mathbb{R}$, and

$$\sum_{x_i \in R_X} p_X(x_i) = 1.$$

Proof. This is immediate from the additive property of the probability measure since

$$\{ \omega \in \Omega \mid X(\omega) \in A \} = \bigcup_{x_i \in A} \{ \omega \in \Omega \mid X(\omega) = x_i \}$$

a disjoint union.

The special cases come from taking $A = (-\infty, a]$, $A = (-\infty, a)$ and $A = \mathbb{R}$, respectively. □

Probability 2 (Year 2)

Mathematics at University

- Weekly hand-in homework during the first year.
- This homework is marked and returned back together with some feedback.
- From second year onwards there are no weekly hand-in homework; *most of the learning process occurs individually*.
- However, exercises are given every week and it relies upon the student to do them or not.
- Solutions to the exercises are posted after some time has passed.

3rd and 4th differences with high-school Maths: at University, *most of the work is autonomous* and *learning is the student's own responsibility*.

- During Year 1 and the first semester of Year 2, all courses are mandatory.
- From the second semester of Year 2 onwards, all courses are optional.

5th difference with high-school Maths: at University, there is *diversity and abundance of topics*.

Mathematics at University

List of Year 3 courses for Mathematics at The University of Manchester:

Fractal Geometry	Mathematical Logic	Problem Solving by Computer
Topology	Applied Complex Analysis	Convex Optimization
Riemannian Geometry	Green's Functions, Integral Equations and Applications	Martingales with Applications to Finance
Group Theory	Viscous Fluid Flow	Markov Processes
Commutative Algebra	Wave Motion	Statistical Inference
Coding Theory	Elasticity	Time Series Analysis
Hyperbolic Geometry	Mathematical Biology	Medical Statistics
Algebraic Geometry	Symmetry in Geometry and Nature	Regression Analysis
Number Theory	Matrix Analysis	Multivariate Statistics and Machine Learning
Combinatorics and Graph Theory	Numerical Analysis II	Generalised Linear Models
		Mathematical Modelling in Finance

David Hilbert (1862-1943) was a German mathematician; he was one of the most important mathematicians of 19th and 20th centuries.

In one his lectures, he proposed the following problem:

Suppose that there exists a hotel with infinitely many rooms numbered 1, 2, 3, 4, 5, . . . and such that each of the rooms is occupied by a single guest.

Now suppose that a new guest comes to the hotel seeking one room to accommodate himself.

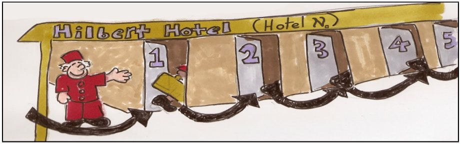
Question: Is it possible to accommodate the new guest in this infinite hotel?

Infinite Sets: Hilbert's Hotel

The answer is yes!

Take the guest in room 1 and move him/her to room 2. But room 2 is occupied, so to free it up for the guest in room 1, move the guest in room 2 to room 3. Since the hotel has infinitely many rooms, we can always free up room number n to let the guest in room number $n - 1$ accommodate him/herself in room n.

In this way we reallocate all the guests in the Hotel leaving room 1 for the new guest.



Infinite Sets: Hilbert's Hotel

This thought experiment shows how different infiniteness can be from finiteness.

We will now go step by step into abstracting the notion of "collection of things" and generalize the structure we've seen with Hilbert's Hotel. We will do this in a way which very closely resembles how Mathematics is done at University.

Infinite Sets: formalizing the ideas

In order to formalize what we have done, we will first abstract the notion of a "family of things". We do this by introducing some definitions.

Definition

A *set* is a well-defined collection of distinct objects, also called the *elements* of the set.

If A is a set, we write $a \in A$ to denote that a is an element of A.

Well-definedness means that the description of the elements of the set is not ambiguous.

Question: Why is *"the set of all big things"* not well-defined?

Infinite Sets: formalizing the ideas

If we know exactly what the elements of a set A are, we write $A = \{a, b, c, \dots\}$, where $a, b, c \in A$.

The following are examples of sets:

- $\{7\}$
- {orange, banana, apple}
- $\{\} = \varnothing$

Infinite Sets: formalizing the ideas

Question: Can you give any other examples of sets that you've seen so far in your mathematical studies?

Some important sets you've worked with again and again are the following:

- $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$; this is the set of natural numbers.
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$; this is the set of integers.
- $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$; this is the set of rational numbers.

Infinite Sets: formalizing the ideas

To formalize the notions of finite and infinite into a proper mathematical concepts we need to be able to compare "sizes" of sets.

The first question to answer here would be: "How do we know if two sets have the same amount of elements in them"?

Definition

Let A and B be sets. We say that *A and B have the same cardinality*, written $|A| = |B|$ if there is a one-to-one correspondence between the elements of A and B.

Exercise: Let A be the set of fingers in your left hand and let B be the set of fingers in your right hand. Show that $|A| = |B|$.

Infinite Sets: formalizing the ideas

Before talking about infinite sets we need to make precise what finite set means!

Definition

Let A be a set. We say that A is *finite* if A has the same cardinality as the set $\{1, 2, 3, \dots, n\}$, for some natural number n. Symbolically, A is finite if there exists some $n \in \mathbb{N}$ such that:

$$|A| = |\{0, 1, 2, 3, \dots, n\}|$$

If A is finite and $|A| = |\{1, 2, 3, \dots, n\}|$ for some $n \in \mathbb{N}$, we say that *the cardinality of A is n*, denoted by $|A| = n$.

We define $|\varnothing| = 0$.

Infinite Sets: formalizing the ideas

And finally the long-awaited definition of an infinite set:

Definition

A set A is *infinite* if it is not finite. This is, A is infinite if there doesn't exist any $n \in \mathbb{N}$ such that

$$|A| = |\{1, 2, 3, \ldots, n\}|$$

A priori we don't know that such objects as infinite sets exist; we have to prove it. In order to do this, we pick one set that we might suspect to be infinite, for example $\mathbb{N}$.

Theorem

The set of all natural numbers, $\mathbb{N}$, is an infinite set.

In order to show this result, we use a proof technique called *Proof by Contradiction*. It consists in assuming first that the result is false; from this, using logical arguments we want to arrive at a statement which contradicts what we assumed. This will mean that our assumption is wrong, so that the original statement must be true.

Infinite Sets: $\mathbb{N}$ is infinite

Proof.

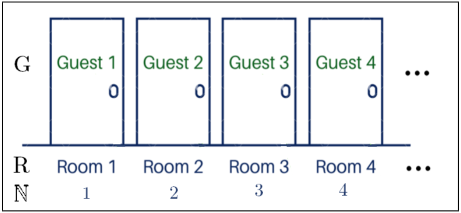
Assume for contradiction that the statement is false, so that $\mathbb{N}$ is not infinite. Then $\mathbb{N}$ is finite, so by definition of finite, $|\mathbb{N}| = |\{1, 2, 3, \ldots, n\}| = n$ for some natural number n. Note that $1, 2, \ldots n$ are all natural numbers, so we thus have that $\mathbb{N} = \{1, 2, 3, \ldots, n\}$.

Moreover, if n is a natural number, then so is $n + 1$, and so $n + 1 \in \mathbb{N}$. But this is a contradiction as if $\mathbb{N} = \{1, 2, 3, \ldots, n\}$ then $n + 1$ is not an element of $\mathbb{N}$. Therefore our assumption was wrong and $\mathbb{N}$ is indeed infinite, as required. □

Infinite Sets: back to Hilbert's Hotel

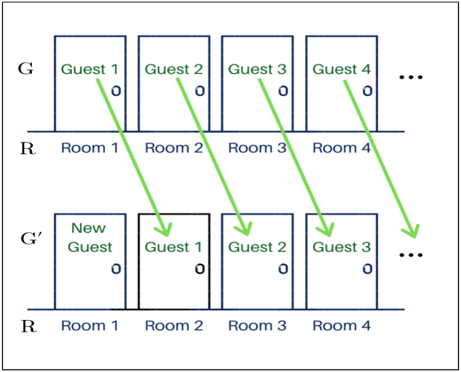
Let's revisit Hilbert's Hotel using our new definitions.

Let R be the set of all rooms in Hilbert's Hotel and let G be the set of all guests in the hotel (recall that there is one guest in each room, so that all rooms are occupied). Note that we have $|R| = |G| = |\mathbb{N}|$:



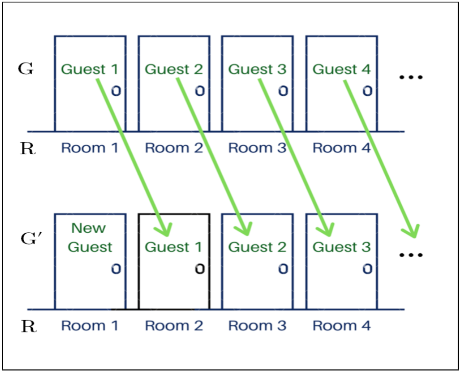
Infinite Sets: back to Hilbert's Hotel

Let now G' be the set of all guests in the hotel together with the new guest; intuitively, we might think that G' has one more element than G, so that G and G' cannot have the same amount of elements, i.e. we might suspect that $|G| \neq |G'|$.



Infinite Sets: back to Hilbert's Hotel

But note that $|G'| = |R|$, and as $|R| = |G|$, it follows that $|G'| = |G|$. In other words "*infinity plus one is still infinity*".



Infinite Sets: yet another property of infinity

Now that we have the technical definitions, we can ask mathematical questions about sets which occur naturally when we do Mathematics. Let us prove the following remarkable statement:

Theorem

Let $\mathbb{N}$ be the set of natural numbers and let $\mathbb{E}$ be the set of even natural numbers. Then $|\mathbb{N}| = |\mathbb{E}|$.

This is another example of our intuition failing in the infinite world; the first thing one would think of is that there are precisely half the number of even natural numbers than there are natural numbers:

$$|\{1, 2, 3, 4, 5, 6, 8, 10 \ldots\}| \stackrel{?}{=} |\{2, 4, 6, 8, 10, \ldots\}|$$

Proof.

We use the definition of two sets having the same cardinality; in order to show that $|\mathbb{N}| = |\mathbb{E}|$ we need to give a one-to-one correspondence between the elements of $\mathbb{N}$ and $\mathbb{E}$.

Question: How can we do this?

Let n be a natural number. Then we can pair-up n with the even natural number $2n$, giving thus the required one-to-one correspondence:

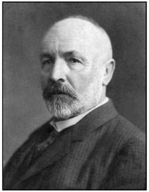
$\mathbb{N}$	1	2	3	$\ldots$	n	$\ldots$
	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$	
$\mathbb{E}$	2	4	6	$\ldots$	$2n$	$\ldots$

□

Infinite Sets: final remarks

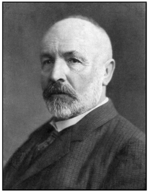
$|\mathbb{N}|$ is so important that mathematicians give this quantity a special name: $\aleph_0$ (read: *aleph zero* or *aleph null*). Here $\aleph$ is the first letter of the Hebrew alphabet.

We could keep asking questions like this; for example, is it the case that $|\mathbb{Q}| = \aleph_0$? Or maybe we also have that $|\mathbb{R}| = \aleph_0$? The former is actually true; the latter was shown to be false in 1891 by another German mathematician, Georg Cantor:



In fact, Cantor showed that *there are some infinities bigger than others*; he even showed that there are an infinite amount of distinct infinities, each one bigger than the other!

With is work, Cantor started *Cardinal Arithmetic*, which is the "arithmetic of infinity"; we have seen an example of it in action with Hilbert's Hotel.



If you decide to do a degree in Mathematics and you like to think about these topics, you can take a course in Set Theory delve more into the mathematical concept of infinity.

4. Throughout this question, work in ZFC (so assume the Axiom of Choice). All arithmetic operations below are those of cardinal arithmetic.

(a) What is meant by saying that κ is a *limit cardinal*? Suppose that κ is a limit cardinal, and suppose that λ is a cardinal with $\lambda \geq \mathrm{cf}\kappa$. Show that

$$\kappa^\lambda = \left(\sup\left\{\mu^\lambda : \mu < \kappa \text{ is a cardinal }\right\}\right)^{\mathrm{cf}\kappa}.$$

[9 marks]

(b) State *Hausdorff's formula*. Show that if $n < \omega$ then

$$\aleph_n^{\aleph_1} = \aleph_n \cdot 2^{\aleph_1}.$$

Show that

$$\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}.$$

[8 marks]

(c) Suppose that $2^{\aleph_1} = \aleph_2$. Show that

$$\aleph_\omega^{\aleph_0} \neq \aleph_{\omega_1}.$$

[3 marks]

The End