INVARIANTS OF THE SIMULTANEOUS CONJUGACY PROBLEM OF MATRICES OVER $\mathbb{C}$

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Abstract. In this paper we establish a way of checking if two d-tuples of $n \times n$ matrices over $\mathbb{C}$ are conjugate using a function $F_d : \mathbb{C}^{dn^2} \to \mathbb{C}^k$ that outputs the coefficients in the polynomials of the reduced Gröbner basis of certain ideal. We give bounds for the $k \in \mathbb{N}$ in the codomain of F_d and we illustrate the solution of the problem by means of small examples implemented in *Singular*.

Contents

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1. Introduction

It is a well-known result in Linear Algebra that two $n \times n$ matrices A and B over $\mathbb{C}$ are conjugate (i.e. $\exists Z \in \mathrm{GL}_n(\mathbb{C})$ such that $Z^{-1}AZ = B$) if and only if A and B have the same Jordan Normal Form, up to re-ordering of Jordan blocks. Our aim with this paper is to obtain a necessary and sufficient condition that characterizes the property of two d-tuples of $n \times n$ matrices over $\mathbb{C}$ being conjugate. In other words, we want to obtain an *invariant* that uniquely determines simultaneous conjugation of matrices over $\mathbb{C}$. This problem has been well known for decades, and there have been various studies establishing some results; for these we refer to [Pro76] and in particular to [Fri83], where an explicit solution is constructed up to a finite number of exceptions.

This paper takes on what is discussed in [KT18, p. 13, Section 2.4] and develops it giving an explicit illustration of the 'wildness' of the classification of d-tuples of $n \times n$ matrices over $\mathbb{C}$ under simultaneous conjugation; we

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denote this equivalence relation on $M_n^d(\mathbb{C})$ by $\sim_d$. The paper is divided in three themes, the first one being where the full explanation of the existence and the explicit description of the invariant are given; this can be found in Section 3. In the first part of this section we exhibit how after fixing $d \in \mathbb{N}$ and expressing $\sim_d$ as a (0-definable) formula $E(\overline{x}, \overline{y})$ in the language of rings, a model theoretic result (Proposition 3.0.1) ensures the existence of a (0-definable) function $F_d : \mathbb{C}^{dn^2} \longrightarrow \mathbb{C}^k$ for some $k \in \mathbb{N}$ such that for $\overline{A}$ and $\overline{B}$ in $M_n^d(\mathbb{C})$, $\overline{A}$ and $\overline{B}$ are simultaneous conjugate if and only if $F_d(\overline{\alpha}) = F_d(\overline{\beta})$, where $\overline{\alpha}$ (respectively $\overline{\beta}$) is a vector in $\mathbb{C}^{dn^2}$ representing the entries of all matrices of $\overline{A}$ (respectively $\overline{B}$). We show that $E(\overline{x}, \overline{y})$ is a finite conjunction of polynomials equal to 0 in the variables $\overline{x}$ and $\overline{y}$ (which correspond to two arbitrary d-tuples of $n \times n$ matrices $\overline{X}$ and $\overline{Y}$, respectively); by specializing $\overline{y}$ to $\overline{\alpha}$ in these polynomials, where $\overline{\alpha}$ represents the entries of the matrices of a concrete tuple $\overline{A} \in M_n^d(\mathbb{C})$, the specialized polynomials give rise to an ideal $J_{\overline{\alpha}}$. From this we explain how to obtain $I_{\overline{\alpha}}$, the vanishing ideal of the Zariski closure of the equivalence class of $\overline{\alpha}$; after a topological remark (Remark 3.0.2) it follows that $F_d(\overline{\alpha})$ is a vector consisting of the coefficients of the polynomials of the reduced Gröbner basis of $I_{\overline{\alpha}}$.

In the second half of Section 3 we follow the idea of the Gröbner Cover developed by Montes in [Mon18] and we modify and apply it to our problem in order to give in subsection 3.2.5 a description of the invariant via a partition of $\mathbb{C}^{dn^2}$.

Section 4 is concerned with obtaining bound for the $k \in \mathbb{N}$ in the codomain of the function F_d given in terms of the length d of the tuple of matrices, the size of the matrices n and the dimension of the ideal $J_{\overline{\alpha}}$ emerging from the solution of the problem. This is achieved by applying known bounds for the cardinality and maximal degree of generators of the reduced Gröbner basis of an ideal (Theorem 4.1.1) and it's radical ([Lap06, p. 193, Algorithm 8]) after fixing a monomial ordering on the variables; having obtained these bounds applied to our context we get an upper bound on the number of entries of $F_d(\overline{\alpha})$ for $\overline{\alpha} \in \mathbb{C}^{dn^2}$, and since all the bounds are independent of the choice of $\overline{\alpha}$ we obtain a bound for k at the end of subsection 4.2.

The last topic of this paper is encapsulated in the three Sections 5, 6 and A; it deals with the implementation of some of the results using the Computer Algebra System (CAS for short) *Singular*. We start delivering the layout of our problem using Singular by giving and explaining some procedures that set-up this CAS for our purposes and then we use these to present worked examples that illustrate the solution to the problem in small cases when $d = 1, 2$ and $n = 2, 3$.

Finally, we also made Section 7 to list a set of open questions that have arisen from this paper and improvements that could be made on the work presented here.

2. Notation and Background Theory

We will start by fixing some notation. Throughout this paper, d will denote the number of matrices in each of the tuples $\overline{X}$ and $\overline{Y}$, and n will be the size of the square matrices in each of the d-tuples above. Being more explicit, we will consider the tuples of matrices $\overline{X} = (X_1, \ldots, X_d)$ and $\overline{Y} = (Y_1, \ldots, Y_d)$, where $X_i, Y_j \in M_n(\mathbb{C})$ for all $i, j \in \{1, \ldots, d\}$. We define the *dimensions of the problem* to be the pair (d, n) for concrete values of d and n.

Unless stated otherwise, we will denote the *variables* by lowercase letters and the different *sets of variables* by overlined lowercase letters, so $\overline{x} = (x_1, \ldots, x_t)$ is a set of t distinct variables. Moreover, if $\overline{x} = (x_1, \ldots, x_t)$ we set $|\overline{x}| = t$. Having this in mind, for a fixed field K we define $K[\overline{x}]$ to be the *polynomial ring* with variables $\overline{x}$ over K. As standard, we write $\deg(f)$ for the maximum of the degrees of monomials with non-zero coefficients in the polynomial $f \in K[\overline{x}]$; we also let $\operatorname{totdeg}(f)$ to be maximal of the sum of all the powers of the variables in a single monomial of f. Throughout the paper, K will be an arbitrary field and $\overline{K}$ will be an algebraically closed extension of K. In practice we will be mostly focused on $\mathbb{C}$, $\mathbb{Q}$ and some simple algebraic and transcendental extensions of $\mathbb{Q}$.

In order to understand the strategy for the solution of the problem, we must first establish some basic notions and results both from Commutative Algebra and Algebraic Geometry. For a complete introduction to these subjects we refer to [CLO15].

2.1. **Basics on Commutative Algebra.** All ideals that we shall consider will be given in the following form:

Definition 2.1.1. A set of polynomials $\{f_1, \ldots, f_s\}$ in $K[\overline{x}]$ is said to *generate the ideal* I if and only if:

$$\forall f \in I,\ f = \sum_{i=1}^{s} g_i f_i,\ \text{for some } g_i \in K[\overline{x}],\ (1 \leq i \leq s).$$

If $\{f_1, \ldots, f_s\}$ generates I, we say that $\{f_1, \ldots, f_s\}$ is a *set of generators* of I and we write $I = \langle f_1, \ldots, f_s \rangle$.

There is a notion of dimension associated to an ideal that we will need for the section on bounds 4. There are many equivalent formulations for the definition of the dimension of an ideal, but we will present here the one that will be more accessible to all readers.

Definition 2.1.2. The dimension of an ideal $I \lhd K[\overline{x}]$ is:

$$\dim(I) = \max\{|\overline{x'}| \ :\ \overline{x'} \subseteq \overline{x},\ I \cap K[\overline{x'}] = \{0\}\}.$$

This coincides with the standard definition of Krull dimension of an ideal I (i.e. the maximal length of chains of prime ideals containing I) if the ground field is algebraically closed. (cf. [CLO15, p. 513, Ex. 4]). The next most important concept that we will make us of throughout this paper is that of a Gröbner basis. However we first have to give the following definition:

Definition 2.1.3. A *monomial ordering* $\succ$ on the set of variables $\overline{x} = (x_1, \ldots, x_t)$ is a partial order that satisfies the following:

(I) $\succ$ is *total*, meaning that for any two monomials m_1 and m_2 in the variables $\overline{x}$ we have that either $m_1 \succ m_2$ or $m_2 \succ m_1$.
(II) For every monomial m in the variables $\overline{x}$, $m \succ 1$.
(III) For every three monomials m, m_1 and m_3 in the variables $\overline{x}$, if $m_1 \succ m_2$, then $mm_1 \succ mm_2$.

A monomial ordering on our set of variables allows us to compare any two monomials. In particular, if we fix a monomial ordering $\succ$ on $\overline{x}$, then we can write any non-zero polynomial $f \in K[\overline{x}]\backslash\{0\}$ in the so called *standard form* (which depends on the choice of $\succ$), meaning that we can write $f = a_1 m_1 + \cdots + a_k m_k$ where $a_i \in K (1 \leq i \leq k)$ and $m_1 \succ \cdots \succ m_k$. In this situation we let:

- $\mathrm{lt}_{\succ}(f) = a_1 m_1$ (the *leading term* of f)
- $\mathrm{lc}_{\succ}(f) = a_1$ (the *leading coefficient* of f)
- $\mathrm{lm}_{\succ}(f) = m_1$ (the *leading monomial* of f)

As an important example of monomial ordering we give the following definition:

Definition 2.1.4. A monomial ordering $\succ$ on $K[x_1, \ldots, x_m]$ is called an *elimination ordering for* $x_1, ..., x_s$ (for some $1 \leq s \leq m$) iff $f \in K[x_1, \ldots, x_s, x_{s+1}, \ldots, x_m]$ and $\mathrm{lm}_{\succ}(f) \in K[x_{s+1}, \ldots, x_m] \implies f \in K[x_{s+1}, \ldots, x_m]$.

2.1.5. As an important example of elimination ordering we point out that for a set of variables $\{x_1, \ldots, x_n\}$ the lexicographic ordering is an elimination ordering for $\{x_1, \ldots, x_j\}$ for $j = 1, \ldots, n$ (cf. [GP08, p. 41]). For further examples of monomial orderings see [CLO15, p. 56-58]].

Having the idea of a monomial ordering we can define:

Definition 2.1.6. The *ideal of leading terms* of an ideal $I \lhd K[\overline{x}]$ is:

$$\mathrm{LT}_{\succ}(I) = \langle \mathrm{lm}_{\succ}(f) : f \in I\backslash\{0\} \rangle.$$

From this we can finally state what do we mean by:

Definition 2.1.7. A *Gröbner basis* of an ideal I is a finite set $G \subseteq I \backslash \{0\}$ such that $\mathrm{LT}_{\succ}(I) = \langle \mathrm{lm}_{\succ}(f) : f \in G \rangle$

To understand now what do we mean by a *reduced* Gröbner basis we have to first introduce the analogue for long division of polynomials for polynomials in several variables. As a preliminary definition we have:

Definition 2.1.8. Let $f, g, h \in K[\overline{x}]$ and write $f = a_1 m_1 + \cdots + a_k m_k$ in standard form with respect to some fixed monomial ordering $\succ$. We say that f *reduces to h modulo g in one step (with respect to $\succ$)* if $\mathrm{lm}(g)$ divides a monomial m_i of f and $h = f - \frac{\mathrm{lt}_{\succ}(f)}{\mathrm{lt}_{\succ}(g)} g$.

Equipped with the one step reduction of polynomials we can then define:

Definition 2.1.9. Let $f, h \in K[\overline{x}]$ and $\mathcal{F} \subseteq K[\overline{x}]$. We say that f *reduces to h modulo $\mathcal{F}$ (with respect to $\succ$)* if h can be obtained from f by a sequence of one-step reductions modulo elements of $\mathcal{F}$. If f cannot be reduced to another polynomial modulo $\mathcal{F}$, we say that f is *reduced modulo $\mathcal{F}$ (with respect to $\succ$)*. If f reduces to r modulo $\mathcal{F}$ and r is reduced modulo $\mathcal{F}$, then r is a *remainder of f modulo $\mathcal{F}$ (with respect to $\succ$).*

For a detailed example of polynomial reduction see [CLO15, p. 67, Example 4]. From this it is then possible to define the reduced Gröbner basis of an ideal.

Definition 2.1.10. A Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_k\}$ of $I \lhd K[\overline{x}]$ is *reduced (with respect to $\succ$)* if each g_i is monic (i.e. $\mathrm{lc}_{\succ}(g_i) = 1$ for each $1 \leq i \leq k$) and reduced modulo every other g_j for $j \neq i$.

Remark 2.1.11. Throughout the paper and after fixing a monomial ordering $\succ$ for our set of variables, we will write G for a Gröbner basis of an ideal and $\mathcal{G}$ for the reduced Gröbner basis of an ideal with respect to $\succ$.

Gröbner bases, both reduced and not reduced, are a very powerful tool in Computational Algebra and Commutative Geometry in general. Here we will list three of their most important and characteristic properties (fixing first a monomial ordering $\succ$ on the variables $\overline{x}$), in particular those that are relevant to our problem:

- Every ideal $I \lhd K[\overline{x}]$ has a Gröbner basis (with respect to $\succ$).
- If G is a Gröbner basis of $I \lhd K[\overline{x}]$, then $I = \langle G \rangle$.
- Every ideal $I \lhd K[\overline{x}]$ has a unique reduced Gröbner basis (with respect to $\succ$) after ordering the polynomials in it according to the order of the leading monomials.

For an in-depth study of Gröbner bases and their applications we refer to the excellent book from Cox, Little and O'Shea [CLO15, p. 49, Chapter 2]. To conclude this subsection, we state the theorem that realises one of the most important applications of Gröbner basis to our project: the *Elimination Theorem.*

Theorem 2.1.12. *Let $I \lhd K[x_1, \ldots x_s, x_{s+1}, \ldots, x_m]$ be an ideal and let $\succ_{elim}$ be an elimination ordering for $x_1, \ldots, x_s$. If G is a Gröbner basis for I, then $G \cap K[x_{s+1}, \ldots, x_m]$ is a Gröbner basis for $I \cap K[x_{s+1}, \ldots, x_m]$.*

Remark 2.1.13. It follows from Theorem 2.1.12 that the same holds if we replace *reduced Gröbner basis* for *Gröbner basis.* Let $\mathcal{G}$ be the reduced Gröbner basis of $I \lhd K[x_1, \ldots x_s, x_{s+1}, \ldots, x_m$ with respect to the elimination ordering $\succ_{\mathrm{elim}}$ for $x_1, \ldots, x_s$. In particular, $\mathcal{G}$ is a Gröbner basis for I, so by Theorem 2.1.12, $\mathcal{G} \cap K[x_{s+1}, \ldots, x_m]$ is a Gröbner basis for $I \cap K[x_{s+1}, \ldots, x_m]$. But $\mathcal{G} \cap K[x_{s+1}, \ldots, x_m]$ is a subset of $\mathcal{G}$ and since $\mathcal{G}$ is reduced it follows (by definition of reduced Gröbner basis) that $\mathcal{G} \cap K[x_{s+1}, \ldots, x_m]$ is also reduced.

Remark 2.1.14. We would like to point out that in a lot of the literature the theorem above is usually stated only when the monomial ordering is the lexicographic one; the theorem stated in this paper is a generalisation of that. For example, in [CLO15, p. 122, Theorem 2] the Elimination Theorem is formulated only for the lexicographic order but the generalised version of the theorem is given as an exercise in this same book in page 128, Exercise 5.

2.2. **Basics on algebraic geometry.** As with the previous section, we start by defining the main object of study in Algebraic Geometry:

Definition 2.2.1. Let $S \subseteq K[x_1, \dots, x_m]$. The *variety* (or *zero-set*) of S is:

$$\mathcal{V}(S) = \{(a_1, \dots, a_m) \in K^m \ : \ f(a_1, \dots, a_m) = 0 \ , \ \forall f \in S\}.$$

Remark 2.2.2. Note that S is just a subset of $K[x_1, \dots, x_m]$; in practice we will only consider varieties of ideals and not arbitrary subsets of our polynomial rings. On the other hand, notice that $\mathcal{V}(S) = \mathcal{V}(I)$, where I is the ideal generated by S.

As a dual notion, we can also associate an ideal to a subset of K^n. Namely:

Definition 2.2.3. Let $V \subseteq K^m$. The *vanishing ideal of* V is:

$$\mathcal{I}(V) = \{f \in K[x_1, \dots, x_m] \ : \ f(\overline{a}) = 0 \ , \ \forall \overline{a} \in V\}.$$

In order to give the main result in this section we also have to introduce the notion of the radical of an ideal.

Definition 2.2.4. Let I be an ideal of a ring R. The *radical* of I is:

$$\sqrt{I} = \{a \in R \ : \ a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

An ideal I is called *radical* if and only if $I = \sqrt{I}$.

Equipped with these definitions we can state the *strong Nullstellensatz*:

Theorem 2.2.5 (Theorem 2 in page 179 of [CLO15])**.** *Let K be an algebraically closed field. Then $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$ for every ideal $I \lhd K[\overline{x}]$.*

2.3. **Basics on model theory.** Throughout this paper we mostly adhere to the notation as presented in [Hod93] and we refer to this book for a concise and comprehensive resource of the subject.

The *alphabet of language* $\mathcal{L}$ is a collection consisting of:

- Logical symbols. These are $\neg$, $\rightarrow$, the quantifier $\forall$, the equality symbol ($\doteq$), brackets, comma and symbols to denote variables.
- Three mutually disjoint sets; the set of relation symbols $\mathcal{R}$, the set of function symbols ($\mathcal{F}$) and the set of constant symbols ($\mathcal{C}$).
- Two maps: $\lambda : \mathcal{R} \longrightarrow \mathbb{N}$ called the *arity of relation symbols*, and $\mu : \mathcal{F} \longrightarrow \mathbb{N}$ called the*arity of function symbols*. We say that $R \in \mathcal{R}$ ($F \in \mathcal{F}$) is n-ary if $\lambda(R) = n$ ($\mu(F) = n$).

From this we can define inductively the $\mathcal{L}$-*terms* starting from the set of variables and constant symbols (cf. [Hod93, p.11]); for $n > 0$, if F is an n-ary function symbol and $t_1, \dots, t_n$ are $\mathcal{L}$-terms, then $F(t_1, \dots, t_n)$ is also an $\mathcal{L}$-term, and nothing else is an $\mathcal{L}$-term.

Define also the *atomic* $\mathcal{L}$-*formulas* to be either $t_1 \doteq t_2$ where t_1 and t_2 are $\mathcal{L}$-terms, or $R(t_1, \dots, t_n)$ where R is an n-ary relation symbol and $t_1, \dots, t_n$ are $\mathcal{L}$-terms. Similarly as with $\mathcal{L}$-terms, we define now inductively the $\mathcal{L}$-formulas; starting with atomic $\mathcal{L}$-formulas, $\mathcal{L}$-formulas are of the form $\neg\varphi$, $\varphi \rightarrow \psi$ and $\forall x \varphi$, where φ and ψ are previously defined $\mathcal{L}$-formulas and x is a variable.

We thus define the *language* $\mathcal{L}$ as the triple consisting of the alphabet of $\mathcal{L}$, the set of all $\mathcal{L}$-terms and the set of all $\mathcal{L}$-formulas.

For us, $\mathcal{L}$ always denotes a first-order language, φ, γ and ψ $\mathcal{L}$-formulas and $\overline{x}, \overline{y}, \overline{z}$ variables as before. In particular, we focus only in the *language of rings* $\mathcal{L} = \{+, -, \cdot, 0, 1\}$, where $+, -$ and $\cdot$ are binary function symbols and 0 and 1 are constants. Moreover we assume all our formulas are in *prenex form*, i.e. they consist of a string of quantifiers followed by a quantifier-free formula. In our case, quantifier-free formulas in the language of rings are simply polynomials.

Having defined the language we can employ the idea of an $\mathcal{L}$-structure (cf. [Hod93, p.2, Section 1.1]) to give the definition of:

Definition 2.3.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure with domain M and let $A \subseteq M$. A subset $S \subseteq M^n$ is called *A-definable* if there is some $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_l)$ and some l-tuple $\overline{a} \in A^l$ such that:

$$S = \{(m_1, \ldots, m_n) \in M^n \ : \ \varphi(m_1, \ldots, m_n, a_1, \ldots, a_l) \text{ is true in } \mathcal{M}\}.$$

If a subset is definable with parameters from $\varnothing$ we say it is 0*-definable*.

We can talk also about definable functions:

Definition 2.3.2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure with domain M. Let $A \subseteq M$ and $S \subseteq M^n$. A function $f : S \longrightarrow M^l$ is called *A-definable in M* if its graph is a definable subset of $M^n \times M^l$.

3. Theoretical Solution of the Problem

In order to proceed we first have to formalize the situation. Fix $d \in \mathbb{N}$ and let $\overline{X}, \overline{Y}$ be two d-tuples of $n \times n$ matrices over $\mathbb{C}$ as defined in 2. We write:

$$\overline{X} \sim_d \overline{Y} \quad \Longleftrightarrow \quad \exists Z \in \mathrm{GL}_n(\mathbb{C}) : Z^{-1}\overline{X}Z = \overline{Y} \tag{1}$$

It is a routine exercise to check that $\sim_d$ is an equivalence relation on the set of d-tuples of $n \times n$ matrices over $\mathbb{C}$ (i.e. on $M_n^d(\mathbb{C})$). In model theoretic terms, we work with the $\mathcal{L}$-structure $(\mathbb{C}, +, -\cdot, 0, 1)$, where $\mathcal{L}$ is the language of rings (cf. 2.3); $\sim_d$ is a 0-definable equivalence relation that can be defined by a formula without parameters $E(\overline{x}, \overline{y})$ expressing $\exists Z \in \mathrm{GL}_n(\mathbb{C}) : Z^{-1}\overline{X}Z = \overline{Y}$. Here, $\overline{x}$ and $\overline{y}$ are dn^2-tuples of variables, where $\overline{x}$ and $\overline{y}$ represent all the entries in all the matrices of $\overline{X}$ and $\overline{Y}$ respectively. Using this language, talking about the equivalence class of a concrete tuple of matrices $\overline{Y} = \overline{A}$ is equivalent to talking about the equivalence class of a concrete tuple $\overline{y} = \overline{\alpha}$, where $\overline{\alpha}$ represents the entries in all the matrices of $\overline{A}$.

The first result that we need to establish is the existence of the invariant that we are looking for. As we will see, the invariant will take form of a function $F_d : \mathbb{C}^{dn^2} \longrightarrow \mathbb{C}^k$ for some $k \in \mathbb{N}$. The existence of such F_d is ensured by the property of the field $\mathbb{C}$ having elimination of imaginaries (cf. [Hod93, Theorem 4.4.6]). We include here the particular result that makes explicit the existence of such function:

Proposition 3.0.1. *Let $\mathcal{L}$ be a language and suppose that T is an $\mathcal{L}$-theory with 2 definable constants. Then T has elimination of imaginaries if and only if for all formulas without parameters $E(\overline{x}, \overline{y})$ of $\mathcal{L}$ where $\overline{x}$ and $\overline{y}$ are of the same length, we have the following: if $E(\overline{x}, \overline{y})$ defines an equivalence relation, then there is a 0-definable function $F : M^{|\overline{x}|} \longrightarrow M^{|\overline{z}|}$ ($M \models T$) for some $\overline{z}$ such that the equivalence classes of $E(M) \subseteq M^{|\overline{x}|} \times M^{|\overline{y}|}$ are exactly the non-empty fibers of F.*

Hence, by 3.0.1, $\mathbb{C}$ having elimination of imaginaries [Hod93, p. 157, Section 4.4] is equivalent to the existence of a 0-definable function $F_d : \mathbb{C}^{dn^2} \longrightarrow \mathbb{C}^k$ for some k such that $\overline{X} \sim_d \overline{Y} \iff F_d(\overline{X}) = F_d(\overline{Y})$, where $\sim_d$ is the 0-definable equivalence relation defined in 1. Our strategy to solve the problem is therefore to find a concrete function F_d whose fibers correspond bijectively to the equivalence classes of $\sim_d$. The non-explicit but formal construction of these functions F_d for $d \in \mathbb{N}$ is given in 3.2.5; quantifier elimination for algebraically closed fields ensures that the 0-definable function F_d will be piecewise rational.

We start by breaking down the expression of $E(\overline{x},\overline{y})$ by noting that it is of the form $\exists\overline{z}\exists w\ V(\overline{x},\overline{y},\overline{z},w)$, where $\overline{z}$ is an n^2-tuple of variables representing the entries in the matrix Z and $V(\overline{x},\overline{y},\overline{z},w)$ is a formula that expresses $Z \in \mathrm{GL}_n(\mathbb{C})$ and $Z^{-1}\overline{X}Z = \overline{Y}$; V stands here for 'variety'. In order to use the Gröbner basis machinery we wish to rewrite $V(\overline{x},\overline{y},\overline{z},w)$ as a finite conjunction of polynomials equal to 0.

First note that $Z \in \mathrm{GL}_n(\mathbb{C})$ can be expressed as $\det(Z) \neq 0$; introducing the variable w we have that the polynomial $w\det(Z) - 1 = 0$ encapsulates that $Z \in \mathrm{GL}_n(\mathbb{C})$. Moreover, it is clear that $Z^{-1}\overline{X}Z = \overline{Y}$ can be rewritten as $\overline{X}Z - Z\overline{Y} = \overline{0}$, where $\overline{0} = (0_{n\times n},\ldots,0_{n\times n})$ is a d-tuple of $n \times n$ zero matrices. After expanding everything out, this last expression will give us dn^2 polynomials $f_i(\overline{x},\overline{y},\overline{z})$ $(1 \leq i \leq dn^2)$ equal to 0.

We therefore see that $V(\overline{x},\overline{y},\overline{z},w)$ is $(w\det(Z)(\overline{z}) - 1 = 0) \wedge (\bigwedge_{i=1}^{dn^2} f_i(\overline{x},\overline{y}) = 0)$, where $\det(Z)(\overline{z})$ indicates that $\det(Z)$ is just a polynomial in the variables $\overline{z}$. Hence the 0-definable formula $E(\overline{x},\overline{y})$ that we started with is

$$\exists\overline{z}\exists w\ \ (w \cdot \det(Z)(\overline{z}) - 1 = 0) \wedge \left(\bigwedge_{i=1}^{dn^2} f_i(\overline{x},\overline{y},\overline{z}) = 0\right).$$

This gives us $dn^2 + 1$ polynomials in $\mathbb{C}[w,\overline{z},\overline{x},\overline{y}]$. Let now $J = \langle f_0, f_1, \ldots, f_{dn^2}\rangle$ be the ideal in $\mathbb{C}[w,\overline{z},\overline{x},\overline{y}]$ generated by $f_0 = w\det(Z) - 1$ and the polynomials $f_i(1 \leq dn^2)$ arising from $\overline{X}Z - Z\overline{Y} = \overline{0}$ as described above. We will call the set of polynomials $\{w\det(Z) - 1, f_1, \ldots f_{dn^2}\} = \{f_0, f_1, \ldots, f_{dn^2}\}$ the *set of initial generators of the ideal J.*

To see why we have introduced the ideal J, let first $\overline{\alpha} \in \mathbb{C}^{dn^2}$ be the tuple representing the entries of the matrices in the concrete tuple $\overline{Y} = \overline{A}$. Consider now the *specialization of* $\overline{y}$ *to* $\overline{\alpha}$, $\sigma_{\overline{\alpha}} : \mathbb{C}[w,\overline{z},\overline{x},\overline{y}] \longrightarrow \mathbb{C}[w,\overline{z},\overline{x}]$ given by $\sigma_{\overline{\alpha}}(f(w,\overline{z},\overline{x},\overline{y})) = f(w,\overline{z},\overline{x},\overline{\alpha})$. By specializing all the initial generators of the ideal J we obtain the ideal $J_{\overline{\alpha}} = \langle\sigma_{\overline{\alpha}}(J)\rangle = \langle\sigma_{\overline{\alpha}}(f_0), \sigma_{\overline{\alpha}}(f_1), \ldots, \sigma_{\overline{\alpha}}(f_{dn^2})\rangle \lhd \mathbb{C}[w,\overline{z},\overline{x}]$.

From here onwards the aim is to find the reduced Gröbner basis of the vanishing ideal of (the Zariski closure of) the equivalence class of $\overline{y} = \overline{\alpha}$ with respect to some fixed monomial ordering $\succ$. We will fix some more notation and we will let $I_{\overline{\alpha}}$ be the vanishing ideal of the equivalence class of $\overline{y} = \overline{\alpha}$. More explicitly, we define:

$$I_{\overline{\alpha}} = \{f \in \mathbb{C}[\overline{x}]\ :\ f(\overline{\beta}) = 0\ ,\ \forall\overline{\beta} \text{ in the Zariski closure of the equiv. class of } \overline{\alpha}\}.$$

Fix now an elimination ordering for $w, \overline{z}$ as defined in 2.1.4. The next task is to find generators for the ideal $H_{\overline{\alpha}} = J_{\overline{\alpha}} \cap \mathbb{C}[\overline{x}]$, whose zero set $\mathcal{V}(H_{\overline{\alpha}})$ is the Zariski closure of the equivalence class of $\overline{y} = \overline{\alpha}$. To do this we eliminate the variables $w, \overline{z}$ from the ideal $J_{\overline{\alpha}}$. Geometrically, this corresponds to taking the Zariski closure of the projection of $\mathcal{V}(J_{\overline{\alpha}})$ onto the coordinates $\overline{x}$ when regarding $\mathcal{V}(J_{\overline{\alpha}})$ as a variety sitting in the complex affine space $\mathbb{C}^{|w|+|\overline{z}|+|\overline{x}|}$, so that $\mathcal{V}(H_{\overline{\alpha}}) = \overline{\pi_{|\overline{x}|}(\mathcal{V}(J_{\overline{\alpha}}))}^{Zar}$, where $\pi_{|\overline{x}|} : \mathbb{C}^{|w|+|\overline{z}|+|\overline{x}|} :\longrightarrow \mathbb{C}^{|\overline{x}|}$ is the projection onto the coordinates $\overline{x}$. Thus:

$\mathcal{V}(H_{\overline{\alpha}})$ $\xleftrightarrow{\text{corresponds to}}$ Zariski closure of the equivalence class of $\overline{A} \in M_n^d(\mathbb{C})$ under $\sim_d$ with entries $\overline{\alpha}$

Remark 3.0.2. Since any equivalence relation on a set S induces a partition on the set S, the varieties corresponding to the different equivalence classes of $M_n^d(\mathbb{C})$ under $\sim_d$ (for explicit correspondence see above) will uniquely determine them (proof below), and so the reduced Gröbner basis of each of the vanishing ideals of these varieties will uniquely determine the equivalence classes of $M_n^d(\mathbb{C})$ under $\sim_d$ (uniqueness follows from the Gröbner bases being reduced).

Before giving the proof of 3.0.2 we need a topological lemma:

Lemma 3.0.3. *Let X be any topological space and let $C \subseteq X$ be a constructible set. Then the frontier $\overline{C}\backslash C$ of C is not dense in $\overline{C}$.*

Proof. We may replace X by $\overline{C}$ and assume that C is dense in X. Write $C = \cup_{i=1}^{n} A_i \cap O_i$ with $n \geq 1$, $O_i \subseteq X$ open and non-empty and $A_i \subseteq X$ closed. Let O be minimal among non-empty intersections of the $O_1, \dots, O_n$. Then for each $i \in \{1, \dots, n\}$ we have $O \subseteq O_i$ or $O_i \cap O = \varnothing$. After a permutation of $\{1, \dots, n\}$ we may assume that $C \cap O = \cup_{i=1}^{k} A_i \cap O$ for some $k \in \{1, \dots, n\}$. It follows that $C \cap O$ is closed in O. On the other hand C is dense in X and O is open in X, which implies that $C \cap O$ is dense in O. Consequently $C \cap O = O$, i.e. $O \subseteq C$. Since $O \neq \varnothing$, the set $X\backslash C$ is not dense in X. □

Proof. We want to show that different equivalence classes of $M_n^d(\mathbb{C})$ under $\sim_d$ have different Zariski closures. Let $E(\overline{x}, \overline{y})$ be the formula defining our equivalence relation and let A and B be two equivalence classes defined by $E(\overline{x}, \overline{\alpha})$ and $E(\overline{x}, \overline{\beta})$, respectively. It suffices to show that the closure map $\overline{}^{Zar} : \mathbb{C}^{|\overline{x}|} \longrightarrow \mathbb{C}^{|\overline{x}|}$ given by $A \mapsto \overline{A}^{Zar}$ is injective. Note that both A and B are quantifier-free definable in the theory of algebraically closed fields, so in particular $E(\overline{x}, \overline{\alpha})$ is a boolean combination of polynomials equal to zero with coefficients given by $\overline{\alpha}$, which in turn implies that A is a boolean combination of Zariski closed sets, hence a constructible set. Suppose now that $\overline{A}^{Zar} = \overline{B}^{Zar}$. By the topological lemma 3.0.3 we know that B is not dense in the frontier $\overline{A}^{Zar}\backslash A$ of A and since $B \subset \overline{A}^{Zar}$, it follows that $B \cap A$, so $A \cap B \neq \varnothing$. But A and B are equivalence classes, so that $A = B$, completing thus the proof. □

In order to find generators for $H_{\overline{\alpha}}$, we need first a Gröbner basis G_1 for $J_{\overline{\alpha}}$; by the Elimination Theorem 2.1.12 the set of polynomials

$$G_2 = G_1 \cap \mathbb{C}[\overline{x}]$$

is a Gröbner basis for $H_{\overline{\alpha}}$ and hence a generating set for $H_{\overline{\alpha}}$. More precisely, due the choice of an elimination ordering, we can write G_2 explicitly as:

$$G_2 = \{f \in G_2 \ : \ \mathrm{lm}(f) \in \mathbb{C}[\overline{x}]\}.$$

As G_2 is a Gröbner basis for $H_{\overline{\alpha}}$, we have that $H_{\overline{\alpha}} = \langle G_2 \rangle$. This is thus the reason for the choice of an elimination ordering and for finding a Gröbner basis G_1 for $J_{\overline{\alpha}}$, as it facilitates the elimination of the variables $w, \overline{z}$ to obtain generators for $H_{\overline{\alpha}}$.

It is now the moment of presenting the punchline of this strategy. Recall that the zero set $\mathcal{V}(H_{\overline{\alpha}})$ of $H_{\overline{\alpha}}$ is precisely the Zariski closure of the equivalence class of $\overline{y} = \overline{\alpha}$ and what we are looking for is the reduced Gröbner basis of the vanishing ideal of this equivalence class, i.e $\mathcal{I}(\mathcal{V}(H_{\overline{\alpha}}))$. As $\mathbb{C}$ is an algebraically closed, by the Nullstellensatz 2.2.5 we have that $\mathcal{I}(\mathcal{V}(H_{\overline{\alpha}})) = \sqrt{H_{\overline{\alpha}}}$, and so finding the reduced Gröbner basis of $I_{\overline{\alpha}}$ is equivalent to finding the reduced Gröbner basis of $\sqrt{H_{\overline{\alpha}}}$.

By Remark 3.0.2 it follows that $F_d(\overline{\alpha})$ (where $F_d : \mathbb{C}^{dn^2} \longrightarrow \mathbb{C}^k$ is the invariant that we are looking for) will be a vector of length k consisting of the coefficients of the polynomials in the reduced Gröbner basis of $I_{\overline{\alpha}} = \sqrt{H_{\overline{\alpha}}}$, i.e. the vanishing ideal of the Zariski closure of the equivalence class of $\overline{\alpha}$.

3.1. **Application of the Gröbner Cover to the solution.** In this subsection we will discuss both the theory of the Gröbner Cover and its application to our situation. For a complete treatise of the topic, we refer to the paper by Montes and Wibmer [MW10] and in particular to Montes' book [Mon18].

Recall from the explanation of the steps given above that we start by choosing a concrete value $\overline{y} = \overline{\alpha}$; what this implies is that the Gröbner basis of $J_{\overline{\alpha}}$, and hence the reduced Gröbner basis of $I_{\overline{\alpha}}$, obviously depend highly on this concrete value. In particular, if we choose a different value $\overline{y} = \overline{\beta}$, then the reduced Gröbner basis for $I_{\overline{\beta}}$ might be completely different from the reduced Gröbner basis of $I_{\overline{\alpha}}$, not only in the coefficients of the polynomials in it but also in the number of polynomials and their "form" ("form" meaning what are exactly these polynomials). Moreover, one of the aims of this paper is to fully describe the function F_d, and so far the procedure described above only gives a recipe to obtain a concrete vector in $\mathbb{C}^k$ that bijectively corresponds to the equivalence class of $\overline{y} = \overline{\alpha}$, namely $F_d(\overline{\alpha})$. The strategy now is to have a stratification of the domain $\mathbb{C}^{dn^2}$ into subsets S_i such that for each S_i and for all $\overline{\alpha} \in S_i$ the reduced Gröbner basis of the ideal $I_{\overline{\alpha}} \triangleleft \mathbb{C}[\overline{x}]$ "looks the same"; this is where the Gröbner Cover comes into play.

To make precise what we mean we start formalizing the situation by letting K be a computable field (e.g. $\mathbb{Q}$) and by fixing a *parametric ideal* $I_{\overline{a}} \triangleleft (K[\overline{a}])[\overline{x}]$, together with a monomial ordering $\succ_{\overline{x}}$ on the variables $\overline{x}$. $(K[\overline{a}])[\overline{x}]$ is the polynomial ring in the variables $\overline{x}$ and coefficients in $K[\overline{a}]$; we will call $\overline{a}$ the *parameters*, which later will take values from the *parameter space* $\overline{K}^{|\overline{a}|}$. We consider $\overline{K}^{|\overline{a}|}$ as a topological space equipped with the K-Zariski topology, so that a subset of $\overline{K}^{|\overline{a}|}$ is closed if and only it is of the form $\mathcal{V}(S)$ for some subset $S \subseteq K[\overline{a}]$. The goal of the Gröbner Cover theory is to fully describe the reduced Gröbner basis of $I_{\overline{\alpha}} = \langle \sigma_{\overline{\alpha}}(I_{\overline{a}}) \rangle$ (with respect to $\succ_{\overline{x}}$) depending on $\overline{\alpha} \in \overline{K}^{|\overline{a}|}$, where $\sigma_{\overline{\alpha}} : (K[\overline{a}])[\overline{x}] \longrightarrow \overline{K}[\overline{x}]$ is the specialization of $\overline{a}$ to $\overline{\alpha}$.

The partition of the parameter space will be given by a special kind of subsets of $\overline{K}^{|\overline{a}|}$, namely:

Definition 3.1.1. A subset S of $\overline{K}^{|\overline{a}|}$ is said to be *locally closed* if and only if it is the intersection of a closed and an open set; equivalently S is locally closed if and only if it is open in its closure.

We will require also the notion of a *regular function* and the closely related *I-regular function*:

Definition 3.1.2. We say that a function $f : S \longrightarrow \overline{K}$ is *regular* if and only if for every $\overline{\alpha} \in S$ there exists an open neighbourhood $U \subset S$ of $\overline{\alpha}$ such that $f(\overline{\beta}) = \frac{p(\overline{\beta})}{q(\overline{\beta})}$ for all $\overline{\beta} \in U$, where $p, q \in K[\overline{a}]$ and $q(\overline{\beta}) \neq 0$ for all $\overline{\beta} \in U$.

Definition 3.1.3. Let S be a locally closed subset of $\overline{K}^{|\overline{a}|}$. We say that a function $f : S \longrightarrow \overline{K}[\overline{x}]$ is *$I_{\overline{a}}$-regular* if and only if for each $\overline{\alpha} \in S$ there exists an open neighbourhood $U \subset S$ of $\overline{\alpha}$ such that $f(\overline{\beta}) = \frac{p(\overline{\beta}, \overline{x})}{q(\overline{\beta})}$ for all $\overline{\beta} \in U$, where $p \in I_{\overline{a}} \triangleleft (K[\overline{a}])[\overline{x}]$ and $q \in K[\overline{a}]$ with $q(\overline{\beta}) \neq 0$ for all $\overline{\beta} \in U$.

It is now time to present how to associate the reduced Gröbner basis of an ideal to each of the subsets that will partition our parameter space. This is realised via the following definition:

Definition 3.1.4. A locally closed subset $S \subset \overline{K}^{|\overline{a}|}$ is called *parametric for $I_{\overline{a}}$ (with respect to $\succ_{\overline{x}}$)* if and only if there exist monic $I_{\overline{a}}$-regular functions $g_1, \ldots, g_r$ such that $\{g_1(\overline{\alpha}, \overline{x}), \ldots, g_r(\overline{\alpha}, \overline{x})\}$ is the reduced Gröbner basis of $I_{\overline{\alpha}}$ for all $\overline{\alpha} \in S$. We then say that $\{g_1(\overline{\alpha}, \overline{x}), \ldots, g_r(\overline{\alpha}, \overline{x})\}$ is *the reduced Gröbner basis of $I_{\overline{\alpha}}$ over S.*

We are now in a position where we can finally define:

Definition 3.1.5. A *Gröbner Cover of $\overline{K}^{|\overline{a}|}$ with respect to $I_{\overline{a}}$ and $\succ_{\overline{x}}$* is a finite set of pairs $\{(S_1, B_1), \ldots (S_r, B_r)\}$ such that:

- For each $1 \leq i \leq r$, S_i is parametric for $I_{\overline{a}}$ and B_i is the reduced Gröbner basis of $I_{\overline{a}}$ over S_i
- $\bigcup_{i=1}^{r} S_i = \overline{K}^{|\overline{a}|}$.

In this situation we say that the S_i are the *segments* of the Gröbner Cover.

Note that in this definition it is not required that the segments of the Gröbner Cover are disjoint, nor it is ensured a priori that such a Gröbner Cover for a fixed ideal and monomial ordering exists, and if it does it doesn't require for it to be unique. However, all this requirements are proven to be satisfied in [Mon18, p.80-86], where it is shown that:

Theorem 3.1.6 (Theorem 5.9 in [Mon18])**.** *If $I_{\overline{a}} \triangleleft (K[\overline{a}])[\overline{x}]$ is homogeneous (with respect to the variables), then there exists a unique Gröbner Cover $\{(S_1, B_1), \ldots (S_r, B_r)\}$ of $\overline{K}^{|\overline{a}|}$ with cardinality r minimal which we call the*

Canonical Gröbner Cover of $\overline{K}^{|\overline{a}|}$ *(with respect to* $I(\overline{a})$ *and* $\succ_{\overline{x}}$*). It is disjoint and two points* $\overline{\alpha}, \overline{\beta} \in \overline{K}^{|\overline{a}|}$ *lie in the same segment if and only if* $LT_{\succ_{\overline{x}}}(I_{\overline{\alpha}}) = LT_{\succ_{\overline{x}}}(I_{\overline{\beta}})$*.*

This theorem can be applied to give a definition of the Canonical Gröbner Cover for arbitrary ideals:

Definition 3.1.7. Let $I_{\overline{a}} \triangleleft (K[\overline{a}])[\overline{x}]$ be an arbitrary ideal and let $J_{\overline{a}} \triangleleft (K[\overline{a}])[\overline{x}, x_0]$ denote its homogenization. By [Mon18, p. 87, Proposition 5.11], the segments of the Canonical Gröbner Cover of $\overline{K}^{|\overline{a}|}$ with respect to $J_{\overline{a}}$ and $\succ_{\overline{x},x_0}$ are parametric with respect to $I_{\overline{a}}$ and $\succ_{\overline{x}}$. The disjoint Gröbner Cover of $\overline{K}^{|\overline{a}|}$ with respect to $I_{\overline{a}}$ and $\succ_{\overline{x}}$ thus obtained will be called the *Canonical Gröbner Cover of* $\overline{K}^{|\overline{a}|}$ *with respect to* $I_{\overline{a}}$ *and* $\succ_{\overline{x}}$*.*

From now onwards we will adapt the same convention as in Montes' book and we will refer to the Canonical Gröbner Cover of our parameter space as simply the Gröbner Cover of our parameter space.

Equipped with the theory we can now apply it to our situation. In order to use the Gröbner Cover machinery we have to translate the problem to the parametric ideal $J_{\overline{a}} \triangleleft (\mathbb{Q}[\overline{a}])[w, \overline{z}, \overline{x}]$; we regard the variables $\overline{y}$ as our parameters $\overline{a}$ which later will take values from the parameter space $\overline{K}^{|\overline{a}|} = \mathbb{C}^{dn^2}$. Fix also an elimination ordering $\succ_{\text{elim}}$ for $w, \overline{z}$ on the variables $w, \overline{z}, \overline{x}$.

We now introduce a modification of Definition 3.1.4, namely:

Definition 3.1.8. A locally closed subset $S \subset \overline{K}^{|\overline{a}|}$ is called *radically parametric for* $I_{\overline{a}}$ *(with respect to* $\succ_{\overline{x}}$*)* if and only if there exist monic $\sqrt{I_{\overline{a}}}$-regular functions $g_1, \ldots, g_r$ such that $\{g_1(\overline{\alpha}, \overline{x}), \ldots, g_r(\overline{\alpha}, \overline{x})\}$ is the reduced Gröbner basis of $\sqrt{I_{\overline{\alpha}}}$ for all $\overline{\alpha} \in S$. We then say that $\{g_1(\overline{\alpha}, \overline{x}), \ldots, g_r(\overline{\alpha}, \overline{x})\}$ is *the reduced Gröbner basis of* $\sqrt{I_{\overline{\alpha}}}$ *over* S*.*

We now can define the radical version of the Gröbner Cover:

Definition 3.1.9. A *radical Gröbner Cover of* $\overline{K}^{|\overline{a}|}$ *with respect to* $I_{\overline{a}}$ *and* $\succ_{\overline{x}}$ is a finite set of pairs of the form $\{(S_1, B_1), \ldots (S_r, B_r)\}$ where:

- For each $1 \leq i \leq r$, S_i is radically parametric for $I_{\overline{a}}$ and B_i is the reduced Gröbner basis of $\sqrt{I_{\overline{a}}}$ over S_i
- $\bigcup_{i=1}^{r} S_i = \overline{K}^{|\overline{a}|}$.

It would be desired to show existence of such a radical Gröbner Cover for a given parametric ideal and monomial ordering. Instead, we use this idea to give the definition of a similar concept:

Definition 3.1.10. A *radical Gröbner Patch* of $\overline{K}^{|\overline{a}|}$ with respect to $I_{\overline{a}}$ and $\succ_{\overline{x}}$ is a finite set of pairs $\{(S_1, B_1), \ldots (S_r, B_r)\}$ such that:

- For each $1 \leq i \leq r$, $B_i = \{g_1(\overline{\alpha}, \overline{x}), \ldots, g_{l(i)}(\overline{\alpha}, \overline{x})\}$ is the reduced Gröbner basis of $\sqrt{I_{\overline{a}}}$ for all $\overline{\alpha} \in S_i$.
- $\bigcup_{i=1}^{r} S_i = \overline{K}^{|\overline{a}|}$.

Note that the difference between the radical Gröbner Cover and the radical Gröbner Patch is that the former requires the polynomials in B_i to be monic $\sqrt{I_{\overline{a}}}$-regular functions, while the latter does not. In the last subsection on the definability of Gröbner Bases 3.2 we show that the function F_d (for fixed d) induces a radical Gröbner Patch for $J_{\overline{a}}$ ensuring thus existence of the radical Gröbner Patch for our situation.

Here is where we take a slightly different approach to the one discussed above when working only with the specific choice $\overline{y} = \overline{\alpha}$. Recall that in the steps needed to obtain the reduced Gröbner basis of $I_{\overline{\alpha}}$ we had to proceed as indicated in the diagram below:

$$J_{\overline{\alpha}} \xrightarrow{\text{elim. var.}} H_{\overline{\alpha}} \xrightarrow{\text{radical}} \sqrt{H_{\overline{\alpha}}}$$

By definition of $H_{\overline{\alpha}}$, this can be re-expressed as:

$$J_{\overline{\alpha}} \xrightarrow{\text{elim. var.}} J_{\overline{\alpha}} \cap \mathbb{C}[\overline{x}] \xrightarrow{\text{radical}} \sqrt{J_{\overline{\alpha}} \cap \mathbb{C}[\overline{x}]}$$

In order to make use or the radical Gröbner Patch to give a full description of the function F_d we will instead consider the following:

$$J_{\overline{\alpha}} \xrightarrow{\text{radical}} \sqrt{J_{\overline{\alpha}}} \xrightarrow{\text{elim. var.}} \sqrt{J_{\overline{\alpha}}} \cap \mathbb{C}[\overline{x}]$$

The key thing to note is that both approaches give the same result; $\mathbb{C}[\overline{x}]$ is trivially radical, so that $\sqrt{J_{\overline{\alpha}}} \cap \mathbb{C}[\overline{x}] = \sqrt{J_{\overline{\alpha}}} \cap \sqrt{\mathbb{C}[\overline{x}]}$, but then it follows from basic properties of radical ideals that $\sqrt{J_{\overline{\alpha}}} \cap \sqrt{\mathbb{C}[\overline{x}]} = \sqrt{J_{\overline{\alpha}} \cap \mathbb{C}[\overline{x}]}$.

Applying this to our problem, what we first do is find a radical Gröbner Patch $\{(S_1, B_1), \dots (S_r, B_r)\}$ of $\mathbb{C}^{dn^2}$ with respect to $J_{\overline{a}}$ and $\succ_{\text{elim}}$; this radical Gröbner Patch gives us a stratification of $\mathbb{C}^{dn^2}$ such that the reduced Gröbner basis of $\sqrt{J_{\overline{\alpha}}} \lhd \mathbb{C}[w, \overline{z}, \overline{x}]$ is $B_i(\overline{\alpha})$ for each $\overline{\alpha} \in S_i$. By the Elimination Theorem 2.1.12 we know that the reduced Gröbner basis of $\sqrt{J_{\overline{\alpha}}} \cap \mathbb{C}[\overline{x}]$ is $B_i(\overline{\alpha}) \cap \mathbb{C}[\overline{x}]$ for each $\overline{\alpha} \in S_i$, and by the argument above we know that this is precisely the reduced Gröbner basis of $I_{\overline{\alpha}} = \sqrt{H_{\overline{\alpha}}}$ for each $\overline{\alpha} \in S_i$.

In terms of how to construct more explicitly our function F_d, we can write it as a piecewise function given by:

$$F_d(\overline{\alpha}) = \begin{cases} f_1(\overline{\alpha}) & \text{if } \overline{\alpha} \in S_1 \\ f_2(\overline{\alpha}) & \text{if } \overline{\alpha} \in S_1 \\ & \vdots \\ f_r(\overline{\alpha}) & \text{if } \overline{\alpha} \in S_r \end{cases}$$

where each $f_i(\overline{\alpha})$ is a k-vector in $\mathbb{C}^k$ (k depending on $\overline{\alpha}$) whose entries are the coefficients of the polynomials of the reduced Gröbner basis $B_i(\overline{\alpha}) \cap \mathbb{C}[\overline{x}]$ of $I_{\overline{\alpha}}$.

3.2. **Definability of Gröbner basis.** Throughout this section, fix $n \in \mathbb{N}$ and a global monomial ordering $\succ$ on $\overline{x}$.

Lemma 3.2.1. *Let $d \in \mathbb{N}$. Then there is some $B \in \mathbb{N}$ such that for each field K and every ideal I of $K[\overline{x}]$ that is generated by d polynomials of total degree at most d, there is a reduced Gröbner basis of I for $\succ$, consisting of at most B polynomials of total degree at most B.*

Proof. From the existence of Gröbner bases, we see that there is some $B \in \mathbb{N}$ (namely, one may choose $B = \max\{\text{totdeg}(g_1), \dots, \text{totdeg}(g_s), s\}$, where $\{g_1, \dots, g_s\}$ is a Gröbner basis for I w.r.t $\succ$) with the required properties for Gröbner bases. By the existence proof for reduced Gröbner bases (cf. [CLO15, p. 93, Theorem 5]) we see that the bound B is also sufficient for our assertion. □

Lemma 3.2.2. *Let $\varphi(\overline{x}, \overline{y})$ be a formula in the language of rings. Then there is some $D \in \mathbb{N}$ such that for all algebraically closed fields K and all $\overline{b} \in K^{|\overline{y}|}$, the ideal of $K[\overline{x}]$ of all polynomials that vanish on $\varphi(K, \overline{b})$ is generated by at most D polynomials of total degree at most D.*

Proof. By model completeness of algebraically closed fields, there are finitely many polynomials $P_i(\overline{x}, \overline{y}, \overline{z})$, $(1 \leq i \leq l)$, with integer coefficients such that φ is equivalent to the formula $\exists \overline{z} \bigwedge_{i=1}^{l} P_i(\overline{x}, \overline{y}, \overline{z}) = 0$. We extend our global monomial order to an elimination order $\succ_{\text{elim}}$ of monomials in variables $\overline{x}, \overline{z}$ for $\overline{z}$. By 3.2.1 applied to the maximal total degree d of the P_i there is some $B \in \mathbb{N}$ such that for each field K and every $\overline{b} \in K^{|\overline{y}|}$, there is a Gröbner basis of the ideal $J_{\overline{b}} = \langle P_i(\overline{x}, \overline{b}, \overline{z}) \rangle_{i=1,\dots,l}$ of $K[\overline{x}, \overline{z}]$ of size at most B all whose polynomials are of total degree at most B. As $\succ_{\text{elim}}$ is an elimination order for $\overline{z}$, the variety defined by $I_{\overline{b}} = J_{\overline{b}} \cap K[\overline{x}]$ is the Zariski closure of $\varphi(K,\bar{b})$ in $K^{|\bar{x}|}$, provided K is algebraically closed (cf. [CLO15, p. 199, Theorem 4] or [GP08, p. 96, Section 1.8.2]). Hence we have degree bounds and a bound on the size of generators of the $I_{\bar{b}}$. But then we also get bounds on number and degrees of generators of $\sqrt{I_{\bar{b}}}$; cf. [GP08, p. 77, Section 1.8.6] (using Rabinowich's trick). Since K is algebraically closed, $\sqrt{I_{\bar{b}}}$ is the ideal of $K[\bar{x}]$ of all polynomials that vanish on $\varphi(K,\bar{b})$. □

Lemma 3.2.3. *Let $d \in \mathbb{N}$ and let $P(\bar{x},\bar{z})$ be the general polynomial over $\mathbb{Z}$ in the variables $\bar{x}$ of degree at most d, i.e.*

$$P(\bar{x},\bar{z}) = \sum_{\deg(\mathfrak{m})\leq d} z_{\mathfrak{m}} \cdot \mathfrak{m}$$

where $\mathfrak{m}$ ranges over monomials in $\bar{x}$ (note that $\bar{z}$ is therefore a tuple of length $\binom{n+d}{n}$, corresponding to the number of monomials in n variables of degree at most d). Then for each $k \in \mathbb{N}$ there is a formula $\beta(\bar{z}_1,\dots,\bar{z}_k)$ in the language of rings such that for all fields K and all $\bar{b}_1,\dots,\bar{b}_k \in K^{|\bar{z}|}$ we have $K \models \beta(\bar{z}_1,\dots,\bar{z}_k)$ if and only if $\mathcal{G} = \{P(\bar{x},\bar{b}_1),\dots,P(\bar{x},\bar{b}_k)\}$ is a reduced Gröbner basis (i.e., the reduced Gröbner basis of the ideal generated by $P(\bar{x},\bar{b}_1),\dots,P(\bar{x},\bar{b}_k)$ for the given global monomial ordering.)

Proof. It is clear that the condition of $\mathrm{lc}_{\succ}(P(\bar{x},\bar{b}_i)) = 1$ for each $1 \leq i \leq k$ is elementary (i.e., it is formulated using only finite first-order logic). Showing that each of the $P(\bar{x},\bar{b}_i)$ are reduced with respect to all other $P(\bar{x},\bar{b}_j)$ $(i \neq j)$ is equivalent (by definition of reduction of polynomials) to showing that for each $P(\bar{x},\bar{b}_i)$, no monomial occurring in $P(\bar{x},\bar{b}_i)$ lies in $\mathrm{LT}(\mathcal{G}\backslash\{P(\bar{x},\bar{b}_i)\})$, but this is elementary because membership in the ideal of leading terms is elementary using the solution of the membership problem for ideals (cf. [DS84, p. 78, item (I)]).

Finally, defining the property of $\mathcal{G}$ being a Gröbner basis in terms of a formula of the language of rings can be done using Buchberger's Criterion (cf. [CLO15, p.86, Theorem 6]). □

Proposition 3.2.4. *Let $\varphi(\bar{x},\bar{y})$ be a formula in the language of rings. Then there are $d,k \in \mathbb{N}$ and a formula $\gamma(\bar{y},\bar{z}_1,\dots,\bar{z}_k)$ in the language of rings such that for the general polynomial $P(\bar{x},\bar{z})$ over $\mathbb{Z}$ in the variables $\bar{x}$ of degree at most d as in 3.2.3, the following hold true in every algebraically closed field K:*

- *For every $\bar{b} \in K^{|\bar{y}|}$ there exist elements $\bar{c}_1,\dots,\bar{c}_k \in K^{|\bar{z}|}$ such that the set $\{P(\bar{x},\bar{c}_1),\dots,P(\bar{x},\bar{c}_k)\}$ is the reduced Gröbner basis for $\succ$ of the ideal of $K[\bar{x}]$ of all polynomials that vanish on $\varphi(K,\bar{b})$.*
- *For all $\bar{b} \in K^{|\bar{y}|}, \bar{c}_1,\dots,\bar{c}_k \in K^{|\bar{z}|}$ we have that $K \models \gamma(\bar{b},\bar{c}_1,\dots,\bar{c}_k)$ if and only if the set of polynomials $\{P(\bar{x},\bar{c}_1),\dots,P(\bar{x},\bar{c}_k)\}$ is the reduced Gröbner basis for $\succ$ of the ideal of $K[\bar{x}]$ of all polynomials that vanish on $\varphi(K,\bar{b})$.*

Proof. Take $D \in \mathbb{N}$ for $\varphi(\bar{x},\bar{y})$ as in 3.2.2. Then take B for n and D as in 3.2.1. Thus we know that for every algebraically closed field K and all $\bar{b} \in K^{|\bar{y}|}$, the ideal of polynomials that vanish on $\varphi(K,\bar{b})$ has a Gröbner basis consisting of B elements, all of total degree at most B. Hence we may take $d = k = B$ and get the first item.

Using the formula $\beta(\bar{z}_1,\dots,\bar{z}_k)$ from 3.2.3 we define γ as required for the second item as follows: By the solution of the ideal membership problem (cf. [DS84, p. 78, item (I)]), there is a formula $\psi(\bar{y},\bar{z}_1,\dots,\bar{z}_k)$ in the language of rings such that in every algebraically closed field K, each $\bar{b} \in K^{|\bar{y}|}$ and all $\bar{c}_1,\dots,\bar{c}_k \in K^{|\bar{z}|}$ we have $K \models \gamma(\bar{b},\bar{c}_1,\dots,\bar{c}_k)$ if and only if the ideal generated by $\{P(\bar{x},\bar{c}_1),\dots,P(\bar{x},\bar{c}_k)\}$ is the ideal generated by all polynomials of degree at most D that vanish on $\varphi(K,\bar{b})$. Now we may take γ as $\beta(\bar{z}_1,\dots,\bar{z}_k)\wedge\psi(\bar{y},\bar{z}_1,\dots,\bar{z}_k)$. □

3.2.5. Coding Zariski closures. Let $\varphi(\bar{x},\bar{y})$ be a formula in the language of rings. Choose $d,k \in \mathbb{N}$, the polynomial $P(\bar{x},\bar{z})$ and a formula γ as in 3.2.4. Let $m = \binom{n+d}{n}$.

Let K be an algebraically closed field and let $F = F_1 \times \cdots \times F_k : K^{|\bar{y}|} \longrightarrow (K^m)^k$ be the following function: Pick $\bar{b} \in K^{|\bar{y}|}$ and take $f_1(\bar{x}),\dots,f_l(\bar{x}) \in K[\bar{x}]$ such that $\{f_1(\bar{x}),\dots,f_l(\bar{x})\}$ is the reduced Gröbner basis of the vanishing ideal of $\varphi(K,\bar{b})$ with $\mathrm{LT}(f_1) \succ \dots \mathrm{LT}(f_l)$. Notice that the f_i are uniquely determined by this requirement, and so by 3.2.4 there are $\bar{c}_1,\dots,\bar{c}_k \in K^{|\bar{z}|}$ with $\{f_1(\bar{x}),\dots,f_l(\bar{x})\} = \{P(\bar{x},\bar{c}_1),\dots,P(\bar{x},\bar{c}_k)\}$.

Then we define $F_i(\bar{b}) \in K^m$ as follows: If $i > l$, then $F_i(\bar{b}) = 0$. If $i \in \{1, \ldots, l\}$, then $F_i(\bar{b})$ lists the coefficients of f_i, indexed by monomials in $\overline{x}$ in the order of monomials as given by $\succ$. It is then clear that for all $\bar{b}_1, \bar{b}_2 \in K^{|\overline{y}|}$ we have $F(\bar{b}_1) = F(\bar{b}_2)$ if and only if $\varphi(K, \bar{b}_1)$ and $\varphi(K, \bar{b}_2)$ have the same Zariski closure.

Using the formula γ it is straightforward to find a formula $\delta(\overline{y}, \overline{v}_1, \ldots, \overline{v}_k)$ in the language of rings, where $\overline{v}_i$ are m-tuples of variables, which defines the graph of the function F in every algebraically closed field K.

We finally can give the proof of the existence of the radical Gröbner Patch 3.1.10 for our problem in hand:

Proof. By quantifier elimination for algebraically closed fields of characteristic 0, we know that every 0-definable function can be written as a piecewise rational map (cf. [Mar02, Proposition 3.2.14]); in particular the function $F : K^{|\overline{y}|} \longrightarrow (K^m)^k$ defined in subsection 3.2.5, which is 0-definable, can be represented as a piecewise rational map. More precisely, there exist 0-definable locally closed sets $S_1, \ldots, S_p \in K^{|\overline{y}|}$ that partition $K^{|\overline{y}|}$ where $S_j = \{R_j = 0 \text{ and } T_j \neq 0 \mid R_j, T_j \in \mathbb{Z}[x_1, \ldots, x_m]\}$ for each $1 \leq j \leq p$, such that for all $\overline{\alpha} \in K^{|\overline{y}|}$, if $\overline{\alpha} \in S_j$, then $f_i(\overline{\alpha}) = \frac{H_i(\overline{\alpha})}{Q_i(\overline{\alpha})}$ for $1 \leq i \leq l$. Therefore, using the description of the f_i given by the general polynomial we have that:

$$F(\overline{\alpha}) = \begin{pmatrix} F_1(\overline{\alpha}) \\ F_2(\overline{\alpha}) \\ \vdots \\ F_k(\overline{\alpha}) \end{pmatrix}$$

where $F_i(\overline{\alpha})$ lists the coefficients of the polynomial

$$\sum_{\deg(\mathfrak{m}) \leq d} F_{i,\mathfrak{m}}(\overline{\alpha}) \cdot \mathfrak{m} = \sum_{\deg(\mathfrak{m}) \leq d} \frac{H_{j,i,\mathfrak{m}}(\overline{\alpha})}{Q_{j,i,\mathfrak{m}}(\overline{\alpha})} \cdot \mathfrak{m}.$$

Here $\frac{H_{j,i,\mathfrak{m}}(\overline{\alpha})}{Q_{j,i,\mathfrak{m}}(\overline{\alpha})}$ indicates that this coefficient depends on $\overline{\alpha}$, the locally closed set S_j to which $\overline{\alpha}$ belongs to, the polynomial f_i and the monomial $\mathfrak{m}$.

It is now left to show that each of the S_j $(1 \leq j \leq p)$, which is a definable set, can be partitioned into a finite disjoint union of locally closed sets. By quantifier elimination for algebraically closed fields, every definable set is a boolean combination of locally closed sets. We shall show first that any boolean combination of locally closed sets is a constructible set and then that constructible sets are finite disjoint union of locally closed sets.

Let $\mathcal{C}$ be the collection of all constructible sets (i.e. the ones which are a finite union of locally closed sets) and $\mathcal{L}$ the collection of all locally closed sets of $K^{|\overline{y}|}$. It is readily verified that $\mathcal{C}$ is a Boolean algebra (as it is closed under complements, finite unions and intersections) and that $\mathcal{L} \subseteq \mathcal{C}$. On the other hand, if the Boolean algebra $\mathcal{A}$ contains $\mathcal{L}$, then it contains the finite union of locally closed sets, so that $\mathcal{C} \subseteq \mathcal{A}$ and so $\mathcal{C}$ is the Boolean algebra generated by $\mathcal{L}$, proving thus the first assertion.

Let now $K = \bigcap_{i=1}^{n}(O_i \cup A_i)$ be a constructible set, where O_i is open and A_i is closed. We show by induction on n that K can be expressed as a disjoint union of locally closed sets. If $n = 1$, note that $K = O_1 \cup A_1 = (O_1 \backslash A_1) \dot{\cup} A_1$, so the base case holds. Suppose that the statement holds for $k \in \mathbb{N}$. Let L_i $(1 \leq i \leq k)$ be locally closed sets and note that for $k+1$ we have:

$$\begin{aligned} K &= (O_0 \cup A_0) \cap \bigcap_{i=1}^{k}(O_i \cup A_i) \\ &= (O_0 \cup A_0) \cap \dot{\bigcup}_{i=1}^{r} L_i \text{ (by inductive hypothesis)} \\ &= \dot{\bigcup}_{i=1}^{r} L_i \cap (O_0 \cup A_0). \end{aligned} \tag{2}$$

Note that $(O_0 \cup A_0)$ is the complement of a locally closed set. To show the second assertion it is enough then to show that for locally closed sets L and L' in a topological space X, $L \cap (X \backslash L')$ is a finite disjoint union of locally closed sets; this together with 2 completes the induction and hence the proof. Indeed, let $L = O \cap A$ and $X \backslash L' = O' \cup A'$ with O, O' open and A, A' closed. Then:

$$\begin{aligned} L \cap (X \backslash L') &= (O \cap A) \cap (O' \cup A') \\ &= (O \cap O' \cap A) \cup (O \cap A \cap A') \\ &= ((O \cap O' \cap A) \cap (X \backslash A')) \dot{\bigcup} (O \cap A \cap A'). \end{aligned}$$

□

4. Bounds

In this section we will explore the upper bounds for the cardinality of the reduced Gröbner basis of $I_{\overline{\alpha}}$ and for the maximal degree of all polynomials in it. The aim is to give an upper bound for $k \in \mathbb{N}$, the natural number determined by the function $F_d : \mathbb{C}^{dn^2} \to \mathbb{C}^k$ that we are looking for. First we will fix some notation.

Definition 4.0.1. Let K be a field and $I \triangleleft K[x_1, \ldots, x_m]$ be an ideal. Define the *degree of the reduced Gröbner basis of I with respect to a monomial ordering $\succ$* to be:

$$\deg(\mathcal{G}_{I,\succ}) := \max\{\deg(f) \ : \ f \text{ is in the red. Gröbner basis } \mathcal{G}_{I,\succ} \text{ of } I \text{ wrt } \succ\}$$

Set also $m := (d+1)n^2 + 1$, i.e. the number of variables in the polynomial ring $\mathbb{C}[w, \overline{z}, \overline{x}]$ over which we will be working with first.

4.1. Upper bound for the degree of the reduced Gröbner basis of $I_{\overline{\alpha}}$.

The bound we present here depends on the dimension of the ideal $J_{\overline{\alpha}}$ as defined in 2.1.2 and on the dimensions of the problem given by the number of variables m. We use the following bound obtained by Mayr and Ritscher in [MR13, p. 92, Theorem 36]:

Theorem 4.1.1. *Let K be an infinite field and fix a monomial ordering $\succ$. Let I be an arbitrary ideal of dimension r in $K[x_1, \ldots, x_m]$ generated by polynomials $f_1, \ldots, f_s$ of degrees $d_1 \geq \cdots \geq d_s$. Then the degree of the reduced Gröbner basis $\mathcal{G}_{I,\succ}$ of I is bounded by:*

$$\deg(\mathcal{G}_{I,\succ}) \leq 2\left(\frac{1}{2}\left((d_1 \ldots d_{m-r})^{2(m-r)} + d_1\right)\right)^{2^r}$$

Adapting the above theorem to our situation, we set $K = \mathbb{Q}$. The reason for this is that in order to bound the maximal degree of generators of $I_{\overline{\alpha}}$ we will later make use of Laplagne's bound, which was obtained by working over the computable field $\mathbb{Q}$; the bound obtained here is therefore applicable to the functions F_d that can be computed using our approach to the problem and not to all the functions whose existence emerge from the theoretical solution of the problem.

We fix also an elimination ordering $\succ_{\text{elim}}$; from now on we will write $\deg(\mathcal{G}_I)$ instead of $\deg(\mathcal{G}_{I,\succ_{\text{elim}}})$ for any ideal I that we will be working with. We start by giving an upper bound for $\deg(\mathcal{G}_{J_{\overline{\alpha}}})$ by observing that all the initial generators of $J_{\overline{\alpha}}$ have degree 2 except the polynomial $w\det(Z)-1$, whose degree depends on the number of variables. More precisely, if we let $f_1 = w\det(Z) - 1$, then $d_1 = n+1$ and $d_2 = d_3 = \cdots = d_{dn^2+1} = 2$, so that by Theorem 4.1.1 we have:

$$\begin{aligned} \deg(\mathcal{G}_{J_{\overline{\alpha}}}) &\leq 2\left(\frac{1}{2}\left(((n+1)d_2 d_3 \ldots d_{m-r})^{2(m-r)} + (n+1)\right)\right)^{2^r} \\ &= 2\left(\frac{1}{2}\left(\left((n+1)2^{m-r-1}\right)^{2(m-r)} + n + 1\right)\right)^{2^r} \qquad (3) \end{aligned}$$

In order to ease the clarity of the following bounds, we simplify the expression 3 further by bounding it as follows:

$$\begin{aligned}
\deg(\mathcal{G}_{J_{\overline{\alpha}}}) &\leq 2\left(\frac{1}{2}\left(\left((n+1)2^{m-r-1}\right)^{2(m-r)}+n+1\right)\right)^{2^r} \\
&\leq 2\left(\frac{1}{2}\left(\left((n+1)^{m-r}\right)^{2(m-r)}+n+1\right)\right)^{2^r} \qquad (4)\\
&= 2\left(\frac{1}{2}\left((n+1)^{2(m-r)^2}+n+1\right)\right)^{2^r} \\
&\leq 2\left(\frac{1}{2}\left((n+1)^{2(m-r)^2}+(n+1)^{2(m-r)^2}\right)\right)^{2^r} \qquad (5)\\
&= 2\left(\frac{1}{2}\left(2(n+1)^{2(m-r)^2}\right)\right)^{2^r} \\
&= 2\left((n+1)^{2(m-r)^2}\right)^{2^r} \\
&= 2(n+1)^{2^{r+1}(m-r)^2}
\end{aligned}$$

Here 4 follows from $2 \leq n+1$ for all $n \geq 1$ and 5 since clearly $n+1 \leq (n+1)^{2(m-r)^2}$. We therefore have that:

$$\deg(\mathcal{G}_{J_{\overline{\alpha}}}) \leq 2(n+1)^{2^{r+1}(m-r)^2} \qquad (6)$$

The next step is bounding $\deg(\mathcal{G}_{H_{\overline{\alpha}}})$. Recall that $H_{\overline{\alpha}}$ was the ideal obtained by eliminating the variables w and $\overline{z}$ from $J_{\overline{\alpha}}$. By 2.1.12 we know that a Gröbner basis for $H_{\overline{\alpha}}$ is contained in $\mathcal{G}_{J_{\overline{\alpha}}}$, so that the maximal degree of a polynomial in a Gröbner basis for $H_{\overline{\alpha}}$ is bounded above by $\deg(\mathcal{G}_{J_{\overline{\alpha}}})$. Since the maximal degree of all polynomials in the reduced Gröbner basis of an ideal is at most the maximal degree of all polynomials in a given Gröbner basis of this same ideal, we conclude that:

$$\deg(\mathcal{G}_{H_{\overline{\alpha}}}) \leq \deg(\mathcal{G}_{J_{\overline{\alpha}}}) \qquad (7)$$

Combining 6 together with 7 he have that:

$$\deg(\mathcal{G}_{H_{\overline{\alpha}}}) \leq 2(n+1)^{2^{r+1}(m-r)^2} \qquad (8)$$

Now we have to obtain an upper bound for the maximal degree of generators for $I_{\overline{\alpha}} = \sqrt{H_{\overline{\alpha}}}$, the radical of $H_{\overline{\alpha}}$. To achieve this we will make use of the bound given by Laplange using the algorithm in [Lap06, p. 193, Algorithm 8] . Laplange showed that for an ideal $I \triangleleft \mathbb{Q}[x_1, \ldots, x_t]$ in a polynomial ring with t variables generated by s polynomials of maximal degree D, the maximal degree of the generators of $\sqrt{I}$ is bounded above by $(sD)^{2^{O(t^2)}}$. Note that in his paper, Laplagne estimated the arithmetic complexity of his algorithm, from which this bound automatically follows; hence the choice of a computable ground field for the polynomial ring.

In our case, $s = |\mathcal{G}_{H_{\overline{\alpha}}}|$, the cardinality of the reduced Gröbner basis of $H_{\overline{\alpha}}$, and $D = \deg(\mathcal{G}_{H_{\overline{\alpha}}})$. Combining with the above, we have that the maximal degree of generators for $I_{\overline{\alpha}}$ is bounded by $(|\mathcal{G}_{H_{\overline{\alpha}}}|\deg(\mathcal{G}_{H_{\overline{\alpha}}}))^{2^{O(m^2)}}$. We point out that the number of variables in this case is dn^2 and not m, since only the variables $\overline{x}$ occur in the polynomials of $H_{\overline{\alpha}}$; the choice of m in the expression instead of dn^2 is only to ease clarity of the bounds (we can do this since $dn^2 \leq m$ by definition of m). Moreover, by the Elimination Theorem 2.1.12 it follows that $|\mathcal{G}_{H_{\overline{\alpha}}}| \leq |\mathcal{G}_{J_{\overline{\alpha}}}|$. Thus,

$$(|\mathcal{G}_{H_{\overline{\alpha}}}|\deg(\mathcal{G}_{H_{\overline{\alpha}}}))^{2^{O(m^2)}} \leq \left(2|\mathcal{G}_{J_{\overline{\alpha}}}|(n+1)^{2^{r+1}(m-r)^2}\right)^{2^{O(m^2)}} \qquad (9)$$

$$
\begin{aligned}
&= (2|\mathcal{G}_{J_{\overline{\alpha}}}|)^{2^{O(m^2)}} \left((n+1)^{2^{r+1}(m-r)^2}\right)^{2^{O(m^2)}} \\
&= (2|\mathcal{G}_{J_{\overline{\alpha}}}|)^{2^{O(m^2)}} \left((n+1)^{2^{r+1+O(m^2)}(m-r)^2}\right) \\
&= (2|\mathcal{G}_{J_{\overline{\alpha}}}|)^{2^{O(m^2)}} \left((n+1)^{2^{O(m^2)}(m-r)^2}\right) \qquad (10) \\
&= \left(2|\mathcal{G}_{J_{\overline{\alpha}}}|(n+1)^{(m-r)^2}\right)^{2^{O(m^2)}}
\end{aligned}
$$

Here the equality in 10 follows from basic properties of the "Big Oh" notation; the dimension r of the ideal $J_{\overline{\alpha}}$ is at most m, so that $r+1+O(m^2) \leq m+1+O(m^2) = O(m^2)$ and therefore $r+1+O(m^2) = O(m^2)$.

Finally, we apply again Theorem 4.1.1 to the ideal $I_{\overline{\alpha}}$. In order to proceed, we note that we require the dimension of the ideal $I_{\overline{\alpha}}$; it is not in our interest to give a bound for $\deg(\mathcal{G}_{I_{\overline{\alpha}}})$ in terms of the dimension of $I_{\overline{\alpha}}$ as it is not straightforward to compute it, so we will use the fact that $\dim(I_{\overline{\alpha}}) \leq dn^2$, where dn^2 is the number of variables in the polynomial ring over which $I_{\overline{\alpha}}$ is defined. We in turn can set $\dim(I_{\overline{\alpha}}) \leq m$ to ease clarity in the bounds, since $dn^2 \leq m$ by definition of m. Setting $D = \left(2|\mathcal{G}_{J_{\overline{\alpha}}}|(n+1)^{(m-r)^2}\right)^{2^{O(m^2)}}$ we have that:

$$
\begin{aligned}
\deg(\mathcal{G}_{I_{\overline{\alpha}}}) &\leq 2\left(\frac{1}{2}\left(\left(D^{dn^2-\dim(I_{\overline{\alpha}})}\right)^{2(dn^2-\dim(I_{\overline{\alpha}}))} + D\right)\right)^{2^m} \\
&\leq 2\left(\frac{1}{2}\left(\left(D^{dn^2}\right)^{2(dn^2)} + D\right)\right)^{2^m} \\
&\leq 2\left(\frac{1}{2}\left((D^m)^{2m} + D\right)\right)^{2^m} \\
&\leq 2\left(\frac{1}{2}\left(D^{2m^2} + D\right)\right)^{2^m} \\
&\leq 2\left(\frac{1}{2}\left(D^{2m^2} + D^{2m^2}\right)\right)^{2^m} \\
&= 2\left(\frac{1}{2}\left(2D^{2m^2}\right)\right)^{2^m} = 2D^{2^{m+1}m^2}
\end{aligned}
$$

So:

$$
\begin{aligned}
\deg(\mathcal{G}_{I_{\overline{\alpha}}}) &\leq 2\left(\left(2|\mathcal{G}_{J_{\overline{\alpha}}}|(n+1)^{(m-r)^2}\right)^{2^{O(m^2)}}\right)^{2^{m+1}m^2} \\
&= 2\left(2|\mathcal{G}_{J_{\overline{\alpha}}}|(n+1)^{(m-r)^2}\right)^{2^{O(m^2)+m+1}m^2} \\
&= 2\left(2|\mathcal{G}_{J_{\overline{\alpha}}}|(n+1)^{(m-r)^2}\right)^{2^{O(m^2)}m^2} \qquad (11)
\end{aligned}
$$

Note that this bound depends on the dimension r of the ideal $J_{\overline{\alpha}}$, the dimensions of the problem given by $m = (d+1)n^2+1$ and the cardinality of $\mathcal{G}_{J_{\overline{\alpha}}}$. We can go further and give an upper bound for $|\mathcal{G}_{J_{\overline{\alpha}}}|$ in terms of the dimensions of the problem. For this we refer to [Lap06, p.194], where it's stated that the general bound for the number of polynomials in a Gröbner basis of an ideal in m variables over $\mathbb{Q}$, generated by s polynomials of maximum degree D is of order $s^{O(1)}D^{2^{O(m)}}$. In our situation, $s = dn^2+1$ and $D = n+1$, so we have the following bound:

$$|\mathcal{G}_{J_{\overline{\alpha}}}| \leq (dn^2+1)^{O(1)}(n+1)^{2^{O(m)}} \tag{12}$$

Combining 11 with 12 we get the upper bound for $\deg(\mathcal{G}_{I_{\overline{\alpha}}})$:

$$\begin{aligned}\deg(\mathcal{G}_{I_{\overline{\alpha}}}) &\leq 2\left(2\left((dn^2+1)^{O(1)}(n+1)^{2^{O(m)}}\right)(n+1)^{(m-r)^2}\right)^{2^{O(m^2)}m^2}\\ &= 2\left(2(dn^2+1)^{O(1)}(n+1)^{2^{O(m)}+(m-r)^2}\right)^{2^{O(m^2)}m^2}\\ &= 2\left(2(dn^2+1)^{O(1)}(n+1)^{1+(m-r)^2}\right)^{2^{O(m^2)}m^2}\end{aligned}$$

From this it is clear that the maximum degree of generators of the reduced Gröbner basis of $I_{\overline{\alpha}}$ is at most of order doubly exponential in the number of initial variables, as expected. Note that this bound depends only on the dimensions of the problem (given by the number of variables) and the dimension of the ideal $J_{\overline{\alpha}}$, but we can give an upper bound only in terms of the dimensions, by bounding above with the same expression but replacing the exponent $(m-r)^2$ by m^2. We must mention that we cannot claim that the bounds obtained here are sharp due to the simplification of expressions with clearer upper bounds. Finally, we also would like to point out that the dimension of the ideal $J_{\overline{\alpha}}$ appears to be n^2; this is verified in computations for small values of the dimensions (d,n) and it would be desired to verify this for all possible values of the dimensions of the problem.

4.2. **Upper bound for the cardinality of the reduced Gröbner basis of $I_{\overline{\alpha}}$.** To start this section we have to refer again to [Lap06, p.194], where it's stated that the general bound for the number of polynomials in a Gröbner basis of an ideal in m variables over $\mathbb{Q}$, generated by s polynomials of maximum degree D is of order $s^{O(1)}D^{2^{O(m)}}$. It is clear that if G is a Gröbner basis of an ideal in a polynomial ring and $\mathcal{G}$ its reduced Gröbner basis, then $|\mathcal{G}| \leq |G|$, so the bound above also applies to reduced Gröbner bases. Note that this bound holds when the ground field of the polynomial ring is $\mathbb{Q}$, therefore the bounds obtained in this section will be again only applicable to the functions F_d that can be computed using our approach to the problem and not to the functions whose existence emerge from the theoretical solution of the problem.

In our case, we start by bounding $|\mathcal{G}_{J_{\overline{\alpha}}}|$, the number of polynomials in the reduced Gröbner basis of $J_{\overline{\alpha}}$. Recall that $J_{\overline{\alpha}}$ is generated by dn^2+1 polynomials and their maximum degree is $n+1$; we therefore obtain the bound:

$$|\mathcal{G}_{J_{\overline{\alpha}}}| \leq (dn^2+1)^{O(1)}(n+1)^{2^{O(m)}}$$

We now want to bound $|\mathcal{G}_{H_{\overline{\alpha}}}|$. Recall that $H_{\overline{\alpha}}$ was the ideal obtained by eliminating the variables w and $\overline{z}$ from $J_{\overline{\alpha}}$. By 2.1.12 we know that a Gröbner basis for $H_{\overline{\alpha}}$ is contained in $\mathcal{G}_{J_{\overline{\alpha}}}$, so that the cardinality of the reduced Gröbner basis of $H_{\overline{\alpha}}$ is at most $|\mathcal{G}_{J_{\overline{\alpha}}}|$, thus we can directly see that

$$|\mathcal{G}_{H_{\overline{\alpha}}}| \leq (dn^2+1)^{O(1)}(n+1)^{2^{O(m)}}$$

We now have to find an upper bound for the cardinality of the reduced Gröbner basis of $I_{\overline{\alpha}} = \sqrt{H_{\overline{\alpha}}}$. To achieve this we will make use again of the bound given by Laplange using the algorithm in [Lap06, p. 193, Algorithm 8] . Laplange showed that for an ideal $I \lhd \mathbb{Q}[x_1,\ldots,x_t]$ in a polynomial ring with t variables generated by s polynomials of maximal degree D, the cardinality of the obtained Gröbner basis of $\sqrt{I}$ by the algorithm is bounded above by $(sD)^{2^{O(t^2)}}$, and so this is also an upper bound for the cardinality of the reduced Gröbner basis of $\sqrt{I}$.

Noting that the polynomials of the reduced Gröbner basis of $H_{\overline{\alpha}}$ generate $H_{\overline{\alpha}}$ and that an upper bound for the maximum degree of polynomials in the reduced Gröbner basis of $H_{\overline{\alpha}}$ is given by 8 we can therefore give the following upper bound for the cardinality of the reduced Gröbner basis of $I_{\overline{\alpha}}$:

$$|\mathcal{G}_{I_{\overline{\alpha}}}| \leq (|\mathcal{G}_{H_{\overline{\alpha}}}|\deg(\mathcal{G}_{H_{\overline{\alpha}}}))^{2^{O((dn^2)^2)}}$$

We see that this quantity is bounded above by 9 since $dn^2 \leq m$ and so it follows that the upper bound obtained for $\deg(\mathcal{G}_{I_{\overline{\alpha}}})$ is also an upper bound for $|\mathcal{G}_{I_{\overline{\alpha}}}|$; it doesn't come as a surprise then that the number of polynomials in $\mathcal{G}_{I_{\overline{\alpha}}}$ is at most double exponential in the number of initial variables.

Combining the bounds obtained for $\deg(\mathcal{G}_{I_{\overline{\alpha}}})$ and for $|\mathcal{G}_{I_{\overline{\alpha}}}|$ we can therefore bound k, recalling that k is the number of coefficients in all polynomials in the reduced Gröbner basis of $I_{\overline{\alpha}}$. Since we can bound the maximum number of polynomials in $\mathcal{G}_{I_{\overline{\alpha}}}$ by the bound obtained for $\deg(\mathcal{G}_{I_{\overline{\alpha}}})$ and as $\deg(\mathcal{G}_{I_{\overline{\alpha}}})$ is the maximum number of coefficients in any polynomial of $\mathcal{G}_{I_{\overline{\alpha}}}$ it follows that we can bound k by:

$$k \leq \left(2\left(2(dn^2+1)^{O(1)}(n+1)^{1+(m-r)^2}\right)^{2^{O(m^2)}m^2}\right)^2$$

5. Singular commands

In this section we exhibit the different procedures and commands used in Singular to produce a concrete output for the solution of the problem, with their corresponding explanation and discussion. As discussed in 3, in order to provide a complete description of the function F_d we have to make use of the (Canonical) radical Gröbner Cover; this is however not yet implemented in Singular nor in any other Computer Algebra System, so it would be desired to develop it in order to fully describe F_d. However we give the Singular implementation and steps needed to compute $F_d(\overline{\alpha})$ hence giving a way of checking computationally if two d-tuples of $n \times n$ matrices over a computable field are conjugate or not.

In order to improve the exposition of ideas in this section, we will append all the Singular code in A.

As stated in the home page of Singular (`https://www.singular.uni-kl.de/`),

> Singular is a Computer Algebra System for polynomial computations, with special emphasis on commutative and non-commutative algebra, algebraic geometry, and singularity theory. It is free and open-source.

One can download the program following this link or if preferred there is the option of trying the terminal online in `https://www.singular.uni-kl.de:8003/`. We want to note that the online server of Singular has 16GB of memory and Intel Xeon CPU, so any code run here will be limited by the characteristics of this machine.

For a complete explanation of the language and options available in Singular we refer to the online manual of this CAS, that can be accessed via `https://www.singular.uni-kl.de/index.php/singular-manual.html`. We also refer to the Greuel and Pfister book [GP08] for more examples and in particular to [GP08, p. 571, Section C] for a concrete and concise introduction to Singular.

The first issue we have to deal with is the naming of the variables. If $d = 1$, we don't work with tuples of matrices and so we name the variables representing the entries of the matrices in the standard way, i.e. x_{ij} will be the variable representing the entry in the ith row and jth column of the matrix X. If $d > 1$ we will use the following convention: x_{kij} will be the variable representing the entry in the ith row and jth column of the kth matrix X_k in the tuple of matrices $\overline{X}$. As an example, if $(d, n) = (2, 3)$ we then have:

$$\overline{X} = (X_1, X_2) = \left(\begin{pmatrix} x_{111} & x_{112} & x_{113} \\ x_{121} & x_{122} & x_{123} \\ x_{131} & x_{132} & x_{133} \end{pmatrix}, \begin{pmatrix} x_{211} & x_{212} & x_{213} \\ x_{221} & x_{222} & x_{223} \\ x_{231} & x_{232} & x_{233} \end{pmatrix}\right)$$

Since the approach we want to take should be applicable for different values of d and n, the first thing we did is to create a Singular procedure called `dimensions` that sets up the polynomial ring over which all the computations will be done and that creates the first ideal J generated by $w\det(Z)-1$ and the polynomials that arise from $\overline{X}Z = Z\overline{Y}$. `dimensions` is a procedure that takes two integer values, d and n, where d is the number of entries in the tuples of matrices and n is the size of the square matrices as defined in 1. For our task in hand, the polynomial ring with which we will be working with in Singular can have as a ground field $\mathbb{Q}$ or any algebraic or transcendental extension of $\mathbb{Q}$ such as $\mathbb{Q}[i]$ or $\mathbb{Q}(\pi)$. From now onwards we restrict ourselves to take $\mathbb{Q}$ as a base field, yet more exotic examples can be found in 6.

As an example, if after defining the procedure `dimensions` we type in the command line of Singular `dimensions(2,2);` the ring over which the computations will be done will be $\mathbb{Q}[w,\overline{z},\overline{x},\overline{y}]$, where the variables are $w,\overline{z},\overline{x},\overline{y}$, with $\overline{z}=(z_{11},z_{12},z_{21},z_{22})$, $\overline{x}=(x_{111},x_{112},x_{121},x_{122},x_{211},x_{212},x_{221},x_{222})$ and $\overline{y}=(y_{111},y_{112},y_{121},y_{122},y_{211},y_{212},y_{221},y_{222})$.

There is also the issue of what monomial ordering to choose for our task, and here we also have two main choices: we can either use any global monomial ordering such as the lexicographic ordering or we can work with an elimination ordering, and both of them having advantages and disadvantages. By choosing a global ordering, we have the possibility of working with the lexicographic ordering; this ordering has the advantage that the resulting (reduced) Gröbner basis for $J_{\overline{\alpha}}$ has very few polynomials compared to an elimination ordering or even any other global monomial ordering. The issue with this choice is that in the step of elimination of variables we are forced to use the built-in function `eliminate` in Singular, which is computationally expensive and it is not the most straightforward way of proceeding with the elimination of variables. The other choice would be an elimination ordering as described in 2.1.4, being an example the lexicographic elimination ordering for the variables $w,\overline{z}$ (note that this is different from the global lexicographic ordering). With this choice of ordering, elimination variables is very simple, but the cardinality of the resulting reduced Gröbner basis is considerably larger than if we choose just the global lexicographic ordering. From now onwards we will choose an elimination ordering when writing the corresponding Singular commands.

Before proceeding to discuss the Singular commands, we recall the tasks to be done in order to find $F_d(\overline{\alpha})$ defined by the coefficients of the reduced Gröbner basis for $I_{\overline{\alpha}}$:

(1) Fix an elimination ordering $\succ_{\text{elim}}$ on our set of variables. Let $J \triangleleft \mathbb{C}[w,\overline{z},\overline{x},\overline{y}]$ be the ideal generated by $w\det(Z)-1$ and the polynomials arising from $\overline{X}Z = Z\overline{Y}$. Choose $\overline{\alpha}\in\mathbb{C}^{dn^2}$ and obtain the ideal $J_{\overline{\alpha}} \triangleleft \mathbb{C}[w,\overline{z},\overline{x}]$ by specializing J via $\overline{\alpha}$.
(2) Compute the reduced Gröbner basis for $J_{\overline{\alpha}}$ in order to proceed with the next step.
(3) Eliminate the variables w and $\overline{z}$ from $J_{\overline{\alpha}}$ to obtain the ideal $H_{\overline{\alpha}} = J_{\overline{\alpha}} \cap \mathbb{C}[\overline{x}]$.
(4) Obtain a reduced Gröbner basis for $I_{\overline{\alpha}} = \sqrt{H_{\overline{\alpha}}}$.
(5) Collect in a vector the coefficients of the polynomials in the obtained reduced Gröbner basis of $I_{\overline{\alpha}}$. This constitutes the *code* that defines the value of the function F_d at $\overline{\alpha}\in\mathbb{C}^{dn^2}$.

5.1. Obtaining $F_d(\overline{\alpha})$ using Singular.

Step 1: *The Set Up.* As explained in the previous paragraphs, in order to start with the implementation of the theory in Singular we make use of the `dimensions` procedure. However, before that we must load the different libraries (this is, sets of procedures) that will enable us to proceed with the next steps. In particular, the first thing that we do is type in the following commands in the Singular terminal:

```
LIB "primdec.lib";
LIB "poly.lib";
LIB "elim.lib";
option(redSB);
```

The first three lines of command load the `primdec.lib`, `poly.lib` and `elim.lib` library files, while the last command forces Singular to always output the reduced Gröbner basis in any computation with ideals after fixing a monomial ordering on the variables.

After this we load our `dimensions` procedure into the terminal; to do this just copy and paste the code in the terminal as it appears in subsection A.1 of the Appendix. We recall that we will restrict ourselves to work with $\mathbb{Q}$ as a ground field and the lexicographic ordering for the variables w and $\overline{z}$ as a global elimination ordering; in case that one wants to change the ground field or the global ordering this has to be done by tweaking some bits of the code in the `dimensions` procedure. Instances of these can be found in the Appendix A together with some worked examples in Section 6.

After having the libraries and the `dimensions` procedure loaded in the terminal, we can start working with concrete values of d and n; choose d and n and type in the terminal:

```
dimensions(d,n);
```

substituting `d` and `n` in the command for the chosen values.

In order to see what is the ring over which the computations are done, type in the terminal `basering;` and to see what are the objects with which you can work with and that are loaded in the terminal type in `listvar();`. After choosing d and n and setting up the ring using the `dimensions(d,n);` command, one should have the following objects loaded in the terminal: the positive integers d and n, the ring r over which one works with, the matrices $Y_1, \ldots, Y_d$ from the tuple $\overline{Y} = (Y_1, \ldots, Y_d)$ and the initial ideal J, the one generated by $w\det(Z) - 1$ and the polynomials arising from $\overline{X}Z - Z\overline{Y} = \overline{0}$.

Step 2: *Choosing $\overline{\alpha}$ and obtaining $J_{\overline{\alpha}}$ and its reduced Gröbner basis.* In this step we choose the $\overline{\alpha}$ in our ground field for which we want to obtain $F_d(\overline{\alpha})$. Recall that $\overline{\alpha}$ is nothing but a vector of length dn^2 representing the entries of all the matrices in the concrete matrix tuple of which we want to know the code for its equivalence class. In this implementation in Singular we will work with the concrete matrix tuple $\overline{A} = (A_1, \ldots, A_d)$ and not with $\overline{\alpha}$; in particular, the next thing to type in the terminal are the matrices $A_1, \ldots, A_d$ of our tuple $\overline{A}$ as follows:

```
matrix A_(1)[n][n]= - , -, ..., -;
matrix A_(2)[n][n]= - , -, ..., -;
```

$\vdots$

```
matrix A_(d-1)[n][n]= - , -, ..., -;
matrix A_(1)[n][n]= - , -, ..., -;
```

Here one has to replace `d` and `n` with the values chosen in Step 1 and the dashes by the entries of each matrix (read as usual from left to right and starting from the top row). For example, if we have chosen $d = 2$ and $n = 3$ and we choose as a concrete matrix tuple

$$\overline{A} = (A_1, A_2) = \left(\begin{pmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & \frac{2}{3} & \frac{5}{8} \\ \frac{13}{21} & \frac{34}{55} & \frac{89}{144} \\ \frac{233}{377} & \frac{610}{987} & \frac{1597}{2584} \end{pmatrix} \right)$$

then the Singular commands would be:

```
matrix A_(1)[3][3]= 1, -2, 3, -4, 5, -6, 7, -8, 9;
matrix A_(2)[3][3]= 1, 2/3, 5/8, 13/21, 34/55, 89/144, 233/377,
     610/987, 1597/2584;
```

We can now substitute the entries of these matrices for the variables $\overline{y}$ (corresponding to the tuple of matrices $\overline{Y}$) in the initial generators of the ideal J obtaining thus the specialized ideal $J_{\overline{\alpha}}$. This is done by copying and pasting in the terminal the commands in subsection A.2; after this step the specialized ideal is named `J_a` and if we type `J_a;` we can see the list of polynomials that constitute the reduced Gröbner basis of $J_{\overline{\alpha}}$ with respect to the chosen monomial ordering.

Step 3: *Eliminate the variables w and $\overline{z}$.* Having the reduced Gröbner basis of $J_{\overline{\alpha}}$ we can apply the Elimination Theorem 2.1.12; this is done by picking the generators that do not contain the variables w and $\overline{z}$. The implementation in Singular of this step is realised via de command:

```
ideal H_a = nselect(J_a, 1..n^2+1);
```

Note that these n^2+1 variables that we want to eliminate from $J_{\overline{\alpha}}$ come first in our definition of elimination ordering (third line in the commands of subsection A.1).

Step 4: *Obtain the reduced Gröbner basis of $I_{\overline{\alpha}}$.* This is the most straightforward step as it only involves computing the radical of $H_{\overline{\alpha}}$. This is done using the command:

```
ideal I_a = groebner(radical(H_a));
```

And we're done; to see what is the reduced Gröbner basis of $I_{\overline{\alpha}}$ with respect to our monomial ordering we just type `I_a;` in the terminal. From here onwards if we want to check if another matrix tuple $\overline{B}$ is in the equivalence class of $\overline{A}$, we proceed with Steps 2, 3 and 4 with this new matrix tuple by replacing any occurrence of `A` and `a` by `B` and `b` respectively in the presented code, and we check if the reduced Gröbner basis of $I_{\overline{\beta}}$ is the same as the reduced Gröbner basis of $I_{\overline{\alpha}}$, where $\overline{\beta}$ is the vector corresponding to the entries of the matrix tuple $\overline{B}$.

6. Worked examples

Here we give a couple of concrete results of the steps described in Section 5 as an illustrative example of the Singular commands. We would like to note that computations for the cases $(d,n)=(1,2)$ and $(d,n)=(2,2)$ are relatively fast and do give a result; however once we have $d=2$ and $n\geq 3$, the computations blow up and so far we haven't obtained any result for these cases. Moreover, it seems that an increase in n has a greater impact on the computer calculations with respect to an increase in d.

6.1. **Examples with $d=1$.** As a first example, we choose $n=2$ and we take $\mathbb{Q}$ as a ground field for our polynomial ring; we check if the matrices

$$A=\begin{pmatrix} \frac{-2}{3} & \frac{6}{5} \\ \frac{-7}{8} & \frac{-62}{8} \end{pmatrix} \text{ and } B=\begin{pmatrix} 0 & \frac{-71}{4} \\ \frac{-34}{9} & \frac{77}{8} \end{pmatrix}$$

are in the same $\sim_d$-equivalence class. As usual, $\overline{\alpha}$ and $\overline{\beta}$ are vectors corresponding to the entries of the matrices A and B respectively. After running all the steps, the outputs are:

```
> I_a;
I_a[1]=60*x_(1)(2)*x_(2)(1)+60*x_(2)(2)^2+505*x_(2)(2)+373
I_a[2]=12*x_(1)(1)+12*x_(2)(2)+101
```

and

```
> I_b;
I_b[1]=72*x_(1)(2)*x_(2)(1)+72*x_(2)(2)^2-693*x_(2)(2)-4828
I_b[2]=8*x_(1)(1)+8*x_(2)(2)-77
```

so that the reduced Gröbner basis of the vanishing ideals $I_{\overline{\alpha}}$ and $I_{\overline{\beta}}$ are

$$\{60x_{12}x_{21} + 60x_{22}^2 + 505x_{22} + 373, 12x_{11} + 12x_{22} + 101\}$$

and

$$\{72x_{12}x_{21} + 72x_{22}^2 - 693x_{22} - 4828, 8x_{11} + 8x_{22} - 77\}$$

respectively, and so clearly $A \not\sim_d B$.

We now give an example where both matrices are trivially in the same $\sim_d$-equivalence class. We choose now $n = 3$ and $\mathbb{Q}(\pi, e)$ as our ground field. Take our matrices to be

$$A = \begin{pmatrix} \frac{-5\pi}{3} & 0 & 0 \\ 0 & 7e & 0 \\ 0 & 0 & \frac{4\pi}{13} \end{pmatrix} \text{ and } B = \begin{pmatrix} 7e & 0 & 0 \\ 0 & \frac{4\pi}{13} & 0 \\ 0 & 0 & \frac{-5\pi}{3} \end{pmatrix}$$

The corresponding outputs for this example are

```
> I_a;
I_a[1]=39*x_(1)(3)*x_(2)(1)^2*x_(3)(2)
-39*x_(1)(3)*x_(2)(1)*x_(2)(2)*x_(3)(1)
+39*x_(1)(3)*x_(2)(1)*x_(3)(1)*x_(3)(3)-39*x_(1)(3)*x_(2)(3)*x_(3)(1)^2
+39*x_(2)(1)*x_(2)(2)*x_(2)(3)*x_(3)(2)
+78*x_(2)(1)*x_(2)(3)*x_(3)(2)*x_(3)(3)
+(53*pi-273*e)*x_(2)(1)*x_(2)(3)*x_(3)(2)+39*x_(2)(1)*x_(3)(3)^3
+(53*pi-273*e)*x_(2)(1)*x_(3)(3)^2
+(-20*pi^2-371*pi*e)*x_(2)(1)*x_(3)(3)+(140*pi^2*e)*x_(2)(1)
-39*x_(2)(2)^2*x_(2)(3)*x_(3)(1)-39*x_(2)(2)*x_(2)(3)*x_(3)(1)*x_(3)(3)
+(-53*pi+273*e)*x_(2)(2)*x_(2)(3)*x_(3)(1)-39*x_(2)(3)^2*x_(3)(1)*x_(3)(2)
-39*x_(2)(3)*x_(3)(1)*x_(3)(3)^2+(-53*pi+273*e)*x_(2)(3)*x_(3)(1)*x_(3)(3)
+(20*pi^2+371*pi*e)*x_(2)(3)*x_(3)(1)
I_a[2]=39*x_(1)(2)*x_(2)(3)*x_(3)(1)+39*x_(1)(3)*x_(2)(1)*x_(3)(2)
-39*x_(1)(3)*x_(2)(2)*x_(3)(1)+39*x_(1)(3)*x_(3)(1)*x_(3)(3)
+39*x_(2)(2)*x_(2)(3)*x_(3)(2)+78*x_(2)(3)*x_(3)(2)*x_(3)(3)
+(53*pi-273*e)*x_(2)(3)*x_(3)(2)+39*x_(3)(3)^3+(53*pi-273*e)*x_(3)(3)^2
+(-20*pi^2-371*pi*e)*x_(3)(3)+(140*pi^2*e)
I_a[3]=39*x_(1)(2)*x_(2)(1)+39*x_(1)(3)*x_(3)(1)+39*x_(2)(2)^2
+39*x_(2)(2)*x_(3)(3)+(53*pi-273*e)*x_(2)(2)+39*x_(2)(3)*x_(3)(2)
+39*x_(3)(3)^2+(53*pi-273*e)*x_(3)(3)+(-20*pi^2-371*pi*e)
I_a[4]=39*x_(1)(1)+39*x_(2)(2)+39*x_(3)(3)+(53*pi-273*e)
```

and

```
> I_b;
I_b[1]=39*x_(1)(3)*x_(2)(1)^2*x_(3)(2)
-39*x_(1)(3)*x_(2)(1)*x_(2)(2)*x_(3)(1)
+39*x_(1)(3)*x_(2)(1)*x_(3)(1)*x_(3)(3)-39*x_(1)(3)*x_(2)(3)*x_(3)(1)^2
+39*x_(2)(1)*x_(2)(2)*x_(2)(3)*x_(3)(2)
```

```
+78*x_(2)(1)*x_(2)(3)*x_(3)(2)*x_(3)(3)
+(53*pi-273*e)*x_(2)(1)*x_(2)(3)*x_(3)(2)+39*x_(2)(1)*x_(3)(3)^3
+(53*pi-273*e)*x_(2)(1)*x_(3)(3)^2+(-20*pi^2-371*pi*e)*x_(2)(1)*x_(3)(3)
+(140*pi^2*e)*x_(2)(1)-39*x_(2)(2)^2*x_(2)(3)*x_(3)(1)
-39*x_(2)(2)*x_(2)(3)*x_(3)(1)*x_(3)(3)
+(-53*pi+273*e)*x_(2)(2)*x_(2)(3)*x_(3)(1)-39*x_(2)(3)^2*x_(3)(1)*x_(3)(2)
-39*x_(2)(3)*x_(3)(1)*x_(3)(3)^2+(-53*pi+273*e)*x_(2)(3)*x_(3)(1)*x_(3)(3)
+(20*pi^2+371*pi*e)*x_(2)(3)*x_(3)(1)
I_b[2]=39*x_(1)(2)*x_(2)(3)*x_(3)(1)+39*x_(1)(3)*x_(2)(1)*x_(3)(2)
-39*x_(1)(3)*x_(2)(2)*x_(3)(1)+39*x_(1)(3)*x_(3)(1)*x_(3)(3)
+39*x_(2)(2)*x_(2)(3)*x_(3)(2)+78*x_(2)(3)*x_(3)(2)*x_(3)(3)
+(53*pi-273*e)*x_(2)(3)*x_(3)(2)+39*x_(3)(3)^3+(53*pi-273*e)*x_(3)(3)^2
+(-20*pi^2-371*pi*e)*x_(3)(3)+(140*pi^2*e)
I_b[3]=39*x_(1)(2)*x_(2)(1)+39*x_(1)(3)*x_(3)(1)+39*x_(2)(2)^2
+39*x_(2)(2)*x_(3)(3)+(53*pi-273*e)*x_(2)(2)+39*x_(2)(3)*x_(3)(2)
+39*x_(3)(3)^2+(53*pi-273*e)*x_(3)(3)+(-20*pi^2-371*pi*e)
I_b[4]=39*x_(1)(1)+39*x_(2)(2)+39*x_(3)(3)+(53*pi-273*e)
```

From this it is easily verified that the reduced Gröbner bases of $I_{\overline{\alpha}}$ and $I_{\overline{\beta}}$ are identical, so that $A \sim_d B$ as expected.

6.2. **Example with $d = 2$.** This is the example where we work with actual tuples of matrices. Choose $n = 2$ and take $\mathbb{Q}$ as ground field for our polynomial ring. Take now the tuples to be:

$$\overline{A} = (A_1, A_2) = \left(\begin{pmatrix} 1 & -2 \\ -4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & \frac{2}{3} \\ \frac{13}{21} & \frac{34}{55} \end{pmatrix} \right)$$

and

$$\overline{B} = (B_1, B_2) = \left(\begin{pmatrix} \frac{-2}{3} & \frac{6}{5} \\ \frac{-7}{8} & \frac{-62}{8} \end{pmatrix}, \begin{pmatrix} 0 & \frac{-71}{4} \\ \frac{-34}{9} & \frac{77}{8} \end{pmatrix} \right).$$

The outputs in this case are:

```
> I_a;
I_a[1]=3465*x_(2)(1)(2)*x_(2)(2)(1)+3465*x_(2)(2)(2)^2-5607*x_(2)(2)(2)+712
I_a[2]=55*x_(2)(1)(1)+55*x_(2)(2)(2)-89
I_a[3]=3465*x_(1)(2)(1)^2*x_(2)(2)(2)^2-5607*x_(1)(2)(1)^2*x_(2)(2)(2)
+712*x_(1)(2)(1)^2-6930*x_(1)(2)(1)*x_(1)(2)(2)*x_(2)(2)(1)*x_(2)(2)(2)
+5607*x_(1)(2)(1)*x_(1)(2)(2)*x_(2)(2)(1)
+20790*x_(1)(2)(1)*x_(2)(2)(1)*x_(2)(2)(2)-32997*x_(1)(2)(1)*x_(2)(2)(1)
+3465*x_(1)(2)(2)^2*x_(2)(2)(1)^2-20790*x_(1)(2)(2)*x_(2)(2)(1)^2
-10395*x_(2)(2)(1)^2
I_a[4]=1155*x_(1)(2)(1)^2*x_(2)(1)(2)
+2310*x_(1)(2)(1)*x_(1)(2)(2)*x_(2)(2)(2)-1869*x_(1)(2)(1)*x_(1)(2)(2)
-6930*x_(1)(2)(1)*x_(2)(2)(2)+10999*x_(1)(2)(1)
-1155*x_(1)(2)(2)^2*x_(2)(2)(1)+6930*x_(1)(2)(2)*x_(2)(2)(1)
+3465*x_(2)(2)(1)
I_a[5]=3465*x_(1)(1)(2)*x_(2)(2)(2)^2-5607*x_(1)(1)(2)*x_(2)(2)(2)
+712*x_(1)(1)(2)-3465*x_(1)(2)(1)*x_(2)(1)(2)^2
-6930*x_(1)(2)(2)*x_(2)(1)(2)*x_(2)(2)(2)+5607*x_(1)(2)(2)*x_(2)(1)(2)
+20790*x_(2)(1)(2)*x_(2)(2)(2)-32997*x_(2)(1)(2)
```

```
I_a[6]=1155*x_(1)(1)(2)*x_(2)(2)(1)+1155*x_(1)(2)(1)*x_(2)(1)(2)
+2310*x_(1)(2)(2)*x_(2)(2)(2)-1869*x_(1)(2)(2)-6930*x_(2)(2)(2)+10999
I_a[7]=x_(1)(1)(2)*x_(1)(2)(1)+x_(1)(2)(2)^2-6*x_(1)(2)(2)-3
I_a[8]=x_(1)(1)(1)+x_(1)(2)(2)-6
```

and

```
I_b[1]=72*x_(2)(1)(2)*x_(2)(2)(1)+72*x_(2)(2)(2)^2-693*x_(2)(2)(2)-4828
I_b[2]=8*x_(2)(1)(1)+8*x_(2)(2)(2)-77
I_b[3]=1440*x_(1)(2)(1)^2*x_(2)(2)(2)^2-13860*x_(1)(2)(1)^2*x_(2)(2)(2)
-96560*x_(1)(2)(1)^2-2880*x_(1)(2)(1)*x_(1)(2)(2)*x_(2)(2)(1)*x_(2)(2)(2)
+13860*x_(1)(2)(1)*x_(1)(2)(2)*x_(2)(2)(1)
-12120*x_(1)(2)(1)*x_(2)(2)(1)*x_(2)(2)(2)+25077*x_(1)(2)(1)*x_(2)(2)(1)
+1440*x_(1)(2)(2)^2*x_(2)(2)(1)^2+12120*x_(1)(2)(2)*x_(2)(2)(1)^2
+8952*x_(2)(2)(1)^2
I_b[4]=480*x_(1)(2)(1)^2*x_(2)(1)(2)+960*x_(1)(2)(1)*x_(1)(2)(2)*x_(2)(2)(2)
-4620*x_(1)(2)(1)*x_(1)(2)(2)+4040*x_(1)(2)(1)*x_(2)(2)(2)-8359*x_(1)(2)(1)
-480*x_(1)(2)(2)^2*x_(2)(2)(1)-4040*x_(1)(2)(2)*x_(2)(2)(1)-2984*x_(2)(2)(1)
I_b[5]=1440*x_(1)(1)(2)*x_(2)(2)(2)^2-13860*x_(1)(1)(2)*x_(2)(2)(2)
-96560*x_(1)(1)(2)-1440*x_(1)(2)(1)*x_(2)(1)(2)^2
-2880*x_(1)(2)(2)*x_(2)(1)(2)*x_(2)(2)(2)+13860*x_(1)(2)(2)*x_(2)(1)(2)
-12120*x_(2)(1)(2)*x_(2)(2)(2)+25077*x_(2)(1)(2)
I_b[6]=480*x_(1)(1)(2)*x_(2)(2)(1)+480*x_(1)(2)(1)*x_(2)(1)(2)
+960*x_(1)(2)(2)*x_(2)(2)(2)-4620*x_(1)(2)(2)+4040*x_(2)(2)(2)-8359
I_b[7]=60*x_(1)(1)(2)*x_(1)(2)(1)+60*x_(1)(2)(2)^2+505*x_(1)(2)(2)+373
I_b[8]=12*x_(1)(1)(1)+12*x_(1)(2)(2)+101
```

From this it is immediately clear that $\overline{A} \not\sim_d \overline{B}$.

7. Open questions

In this last section we collect the questions that have arisen during this project together with some points where improvements or work can be done. We start by the ones coming from the theoretical side of the paper:

- **Is $J_{\overline{\alpha}}$ already radical for all $\overline{\alpha}$?** If this is the case, we can directly implement the Gröbner Cover to the solution of our problem instead of using the radical Gröbner Cover and the radical Gröbner Patch.
- **General existence of the radical Gröbner Cover**. With the existence of the radical Gröbner Cover we don't require the radical Gröbner Patch. Moreover a canonical version of the radical Gröbner Cover (in case of existing) would also be desired.
- **Improvements of the obtained bounds.**

On the implementation of the solution in CAS we have:

- **Implementation of the radical Gröbner Cover (if existing) in Singular.** In case that this is done, it will be technically possible to give a complete description of F_d.
- **Optimization of the Singular code.**
- **Using other CAS to implement the solution of the problem.**

Appendix A. Singular Code

Here we present the Singular code that enables us to develop some of the theory described in this paper computationally. In order to make use of it, it suffices to copy and paste the full code without any changes. If one has to slightly modify the code given here, we recommend copying and pasting it in a Notepad, changing it there and then copying and pasting the Notepad modified version of the code to the Singular terminal.

A.1. **The `dimensions` procedure.** The code that follows this paragraph is the one corresponding to the case when the ground field of our polynomial ring is $\mathbb{Q}$ and when the monomial ordering is the lexicographic ordering for the variables w and $\overline{z}$, an elimination ordering. After that we explain how to change this code in order to have other choices for the ground field and monomial ordering.

```
proc dimensions(int d, int n)
 {if (d==1)
  {ring r=0,(w, z_(1..n)(1..n), x_(1..n)(1..n), y_(1..n)(1..n)),
  (lp(n^2+1),lp);
  setring r;
  export(r);
  export(d);
  export(n);

  int i;
  int j;
  int k;
  int c;

  matrix Z[n][n];
  for (i=1; i<=n; i++)
   {for (j=1; j<=n; j++) {Z[i,j]=z_(i)(j);};}

  matrix X[n][n];
  for (i=1; i<=n; i++)
   {for (j=1; j<=n; j++) {X[i,j]=x_(i)(j);};}

  matrix Y[n][n];
  for (i=1; i<=n; i++)
   {for (j=1; j<=n; j++) {Y[i,j]=y_(i)(j);};}
  export(Y);

  ideal E = ideal(X*Z-Z*Y);
  ideal J = w*det(Z)-1, E;
  export(J);}
 else
  {ring r= 0,(w, z_(1..n)(1..n), x_(1..d)(1..n)(1..n),
   y_(1..d)(1..n)(1..n)),(lp(n^2+1),lp);
  export(r);
  setring r;
  export(n);
  export(d);

  int i;
  int j;
  int k;
  int c;
```

```
matrix Z[n][n];
for (i=1; i<=n; i++)
 {for (j=1; j<=n; j++) {Z[i,j]=z_(i)(j);};}

for (k=1; k<=d; k++)
 {matrix X_(k)[n][n];
 for (i=1; i<=n; i++)
  {for (j=1; j<=n; j++) {X_(k)[i,j]=x_(k)(i)(j);};};}

for (k=1; k<=d; k++)
 {matrix Y_(k)[n][n];
 for (i=1; i<=n; i++)
  {for (j=1; j<=n; j++) {Y_(k)[i,j]=y_(k)(i)(j);};}
 export (Y_(k));}

for (k=1; k<=d; k++) {ideal E_(k)= ideal(X_(k)*Z-Z*Y_(k));}
for (k=1; k<=d; k++) {ideal J = w*det(Z)-1, E_(1..k);}
export(J);}}
```

A.1.1. *For an algebraic extension of* $\mathbb{Q}$. In order to have an algebraic extension of the rationals as the ground field we first have to change the lines 3 and 26 for

```
{ring r=(0, i),(w, z_(1..n)(1..n), x_(1..n)(1..n), y_(1..n)(1..n)),
(lp(n^2+1),lp);
```

and

```
{ring r=(0, i) ,(w, z_(1..n)(1..n), x_(1..d)(1..n)(1..n),
 y_(1..d)(1..n)(1..n)),(lp(n^2+1),lp);
```

respectively, where here `i` will denote the imaginary unit $i = \sqrt{-1}$. Moreover, after each of these lines we will have to add the minimal polynomial of which the primitive element of the extension is a root of; in our case, the minimal polynomial is $x^2 + 1$ and the line of code that we have to add after the new lines 3 and 26 is

```
minpoly = i^2+1;
```

For the sake of completeness we include the full code for `dimensions(d,n);` if we want the ground field to be $\mathbb{Q}[i]$:

```
proc dimensions(int d, int n)
{if (d==1)
{ring r=(0,i),(w, z_(1..n)(1..n), x_(1..n)(1..n), y_(1..n)(1..n)),
(lp(n^2+1),lp);
minpoly = i^2+1;
setring r;
export(r);
export(d);
export(n);

int i;
```

```
int j;
int k;
int c;

matrix Z[n][n];
for (i=1; i<=n; i++)
{for (j=1; j<=n; j++) {Z[i,j]=z_(i)(j);};}

matrix X[n][n];
for (i=1; i<=n; i++)
{for (j=1; j<=n; j++) {X[i,j]=x_(i)(j);};}

matrix Y[n][n];
for (i=1; i<=n; i++)
{for (j=1; j<=n; j++) {Y[i,j]=y_(i)(j);};}
export(Y);

ideal E = ideal(X*Z-Z*Y);
ideal J = w*det(Z)-1, E;
export(J);}
else
{ring r= (0,i), (w, z_(1..n)(1..n), x_(1..d)(1..n)(1..n),
 y_(1..d)(1..n)(1..n)),(lp(n^2+1),lp);
minpoly = i^2+1;
export(r);
setring r;
export(n);
export(d);

int i;
int j;
int k;
int c;

matrix Z[n][n];
for (i=1; i<=n; i++)
{for (j=1; j<=n; j++) {Z[i,j]=z_(i)(j);};}

for (k=1; k<=d; k++)
{matrix X_(k)[n][n];
for (i=1; i<=n; i++)
{for (j=1; j<=n; j++) {X_(k)[i,j]=x_(k)(i)(j);};};}

for (k=1; k<=d; k++)
{matrix Y_(k)[n][n];
for (i=1; i<=n; i++)
{for (j=1; j<=n; j++) {Y_(k)[i,j]=y_(k)(i)(j);};}
export (Y_(k));}

for (k=1; k<=d; k++) {ideal E_(k)= ideal(X_(k)*Z-Z*Y_(k));}
for (k=1; k<=d; k++) {ideal J = w*det(Z)-1, E_(1..k);}
export(J);}}
```

A.1.2. *For a transcendental extension of* $\mathbb{Q}$. In order to have a transcendental extension of the rationals as the ground field we only have to change the lines 3 and 26 for

```
{ring r=(0, pi),(w, z_(1..n)(1..n), x_(1..n)(1..n),
 y_(1..n)(1..n)),(lp(n^2+1),lp);
```

and

```
{ring r=(0, pi),(w, z_(1..n)(1..n), x_(1..d)(1..n)(1..n),
 y_(1..d)(1..n)(1..n)),(lp(n^2+1),lp);
```

respectively. Here `pi` is π and thus the resulting ground field would be $\mathbb{Q}(\pi)$, but we could have chosen any other letter/word such as `x` representing a variable x, in which case the ground field of the polynomial ring would be $\mathbb{Q}(x)$. Unlike with algebraic extensions, Singular allows us to extend $\mathbb{Q}$ with multiple transcendental numbers over $\mathbb{Q}$, so we could also work over a ground field such as $Q(\pi, x)$ by replacing lines 3 and 26 by

```
{ring r=(0, pi, x),(w, z_(1..n)(1..n), x_(1..n)(1..n),
 y_(1..n)(1..n)),(lp(n^2+1),lp);
```

and

```
{ring r=(0, pi, x),(w, z_(1..n)(1..n), x_(1..d)(1..n)(1..n),
 y_(1..d)(1..n)(1..n)),(lp(n^2+1),lp);
```

respectively.

A.1.3. *Changing the monomial order.* The monomial ordering is given in the last part of lines 3 and 26 of the `dimensions` code. The different options for a monomial ordering in Singular can be found in https://www.singular.uni-kl.de/Manual/4-0-2/sing_31.htm. For example, if we choose the product degree reverse lexicographic ordering as an elimination ordering, the codes for lines 3 and 26 would be

```
{ring r=0,(w, z_(1..n)(1..n), x_(1..n)(1..n), y_(1..n)(1..n)),
(dp(n^2+1),dp);
```

and

```
{ring r=0,(w, z_(1..n)(1..n), x_(1..d)(1..n)(1..n),
 y_(1..d)(1..n)(1..n)),(dp(n^2+1),dp);
```

respectively.

A.2. **Obtaining** $J_{\overline{\alpha}}$**.** Here we present the commands needed to get the specialized ideal $J_{\overline{\alpha}}$. When we want to see if two tuples of matrices are in the same $\sim_d$-equivalence class, we will also need to construct a new specialized ideal $J_{\overline{\alpha}}$ corresponding to the second tuple of matrices; as convention, our tuples are always called $\overline{A}$ and $\overline{B}$ and $\overline{\alpha}$ and $\overline{\beta}$are the vectors representing the entries of these tuples respectively. In order to construct $J_{\overline{\beta}}$ we only need to replace every occurrence of `A` for `B` and every occurrence of `a` for `b`; for the sake of convenience we also append here the code corresponding to obtain $J_{\overline{\beta}}$.

For $J_{\overline{\alpha}}$ we have:

```
if (d==1)
 {int k;
 int i;
 int j;
 for (k=1; k<=size(J); k++)
  {poly f_(k)(1)(1) = subst(J[k], Y[1,1], A[1,1]);
  for (i=2; i<=n; i++)
   {for (j=2; j<=n; j++) {poly f_(k)(i-1)(j) = subst(f_(k)(i-1)(j-1),
    Y[i-1,j], A[i-1,j]);};
    poly f_(k)(i)(1)= subst(f_(k)(i-1)(n), Y[i,1], A[i,1]);};
    for (j=2; j<=n; j++) {poly f_(k)(n)(j) = subst(f_(k)(n)(j-1),
     Y[n,j], A[n,j]);};
     poly f_(k) = f_(k)(n)(n);}
     for (k=1; k<=size(J); k++)
      {for (i=1; i<=n; i++)
       {for (j=1; j<=n; j++){kill(f_(k)(i)(j));}}};}
else
 {int k;
 int i;
 int j;
 int c;
 for (k=1; k<=size(J); k++)
  {for (c=1; c<=d; c++)
   {if (c==1)
    {poly f_(k)(c)(1)(1) = subst(J[k], Y_(c)[1,1], A_(c)[1,1]);
    for (i=2; i<=n; i++)
     {for (j=2; j<=n; j++)
      {poly f_(k)(c)(i-1)(j) = subst(f_(k)(c)(i-1)(j-1),
       Y_(c)[i-1,j], A_(c)[i-1,j]);};
      poly f_(k)(c)(i)(1)= subst(f_(k)(c)(i-1)(n), Y_(c)[i,1],
       A_(c)[i,1]);};
      for (j=2; j<=n; j++) {poly f_(k)(c)(n)(j) =
      subst(f_(k)(c)(n)(j-1), Y_(c)[n,j], A_(c)[n,j]);};
       poly f_(k)(c+1)(1)(1) = subst(f_(k)(c)(n)(n),
       Y_(c+1)[1,1], A_(c+1)[1,1]);}
   else
    {if (c!=d)
     {for (i=2; i<=n; i++)
      {for (j=2; j<=n; j++)
       {poly f_(k)(c)(i-1)(j) = subst(f_(k)(c)(i-1)(j-1),
        Y_(c)[i-1,j], A_(c)[i-1,j]);};
       poly f_(k)(c)(i)(1)= subst(f_(k)(c)(i-1)(n),
        Y_(c)[i,1], A_(c)[i,1]);};
        for (j=2; j<=n; j++) {poly f_(k)(c)(n)(j) =
         subst(f_(k)(c)(n)(j-1), Y_(c)[n,j], A_(c)[n,j]);};
         poly f_(k)(c+1)(1)(1) = subst(f_(k)(c)(n)(n),
          Y_(c+1)[1,1], A_(c+1)[1,1]);}}
    if (c==d)
     {for (i=2; i<=n; i++)
      {for (j=2; j<=n; j++)
       {poly f_(k)(c)(i-1)(j) = subst(f_(k)(c)(i-1)(j-1),
        Y_(c)[i-1,j], A_(c)[i-1,j]);};
       poly f_(k)(c)(i)(1)= subst(f_(k)(c)(i-1)(n),
        Y_(c)[i,1], A_(c)[i,1]);};
       for (j=2; j<=n; j++) {poly f_(k)(c)(n)(j) =
```

```
        subst(f_(k)(c)(n)(j-1), Y_(c)[n,j], A_(c)[n,j]);};
        poly f_(k)=f_(k)(c)(n)(n);}}}
        for (c=1; c<=d; c++)
         {for (k=1; k<=size(J); k++)
           {for (i=1; i<=n; i++)
            {for (j=1; j<=n; j++){kill(f_(k)(c)(i)(j));}}}};};
 int s = size(J);
 ideal Ja = f_(1..s);
 ideal J_a= groebner(Ja);
 kill(Ja);
```

For $J_{\overline{\beta}}$ we have:

```
if (d==1)
{int k;
int i;
int j;
for (k=1; k<=size(J); k++)
{poly f_(k)(1)(1) = subst(J[k], Y[1,1], B[1,1]);
for (i=2; i<=n; i++)
{for (j=2; j<=n; j++) {poly f_(k)(i-1)(j) =
 subst(f_(k)(i-1)(j-1),
 Y[i-1,j], B[i-1,j]);};
poly f_(k)(i)(1)= subst(f_(k)(i-1)(n), Y[i,1], B[i,1]);};
for (j=2; j<=n; j++) {poly f_(k)(n)(j) = subst(f_(k)(n)(j-1),
 Y[n,j], B[n,j]);};
poly f_(k) = f_(k)(n)(n);}
for (k=1; k<=size(J); k++)
{for (i=1; i<=n; i++)
{for (j=1; j<=n; j++){kill(f_(k)(i)(j));}}};}
else
{int k;
int i;
int j;
int c;
for (k=1; k<=size(J); k++)
{for (c=1; c<=d; c++)
{if (c==1)
{poly f_(k)(c)(1)(1) = subst(J[k], Y_(c)[1,1], B_(c)[1,1]);
for (i=2; i<=n; i++)
{for (j=2; j<=n; j++)
{poly f_(k)(c)(i-1)(j) = subst(f_(k)(c)(i-1)(j-1),
 Y_(c)[i-1,j], B_(c)[i-1,j]);};
poly f_(k)(c)(i)(1)= subst(f_(k)(c)(i-1)(n),
Y_(c)[i,1], B_(c)[i,1]);};
for (j=2; j<=n; j++) {poly f_(k)(c)(n)(j) =
 subst(f_(k)(c)(n)(j-1), Y_(c)[n,j], B_(c)[n,j]);};
poly f_(k)(c+1)(1)(1) = subst(f_(k)(c)(n)(n),
 Y_(c+1)[1,1], B_(c+1)[1,1]);}
else
{if (c!=d)
{for (i=2; i<=n; i++)
{for (j=2; j<=n; j++)
{poly f_(k)(c)(i-1)(j) = subst(f_(k)(c)(i-1)(j-1),
 Y_(c)[i-1,j], B_(c)[i-1,j]);};
```

```
poly f_(k)(c)(i)(1)= subst(f_(k)(c)(i-1)(n),
 Y_(c)[i,1], B_(c)[i,1]);};
for (j=2; j<=n; j++) {poly f_(k)(c)(n)(j) =
subst(f_(k)(c)(n)(j-1), Y_(c)[n,j], B_(c)[n,j]);};
poly f_(k)(c+1)(1)(1) = subst(f_(k)(c)(n)(n),
Y_(c+1)[1,1], B_(c+1)[1,1]);}}
if (c==d)
{for (i=2; i<=n; i++)
{for (j=2; j<=n; j++)
{poly f_(k)(c)(i-1)(j) = subst(f_(k)(c)(i-1)(j-1),
 Y_(c)[i-1,j], B_(c)[i-1,j]);};
poly f_(k)(c)(i)(1)= subst(f_(k)(c)(i-1)(n),
 Y_(c)[i,1], B_(c)[i,1]);};
for (j=2; j<=n; j++) {poly f_(k)(c)(n)(j) =
 subst(f_(k)(c)(n)(j-1), Y_(c)[n,j], B_(c)[n,j]);};
poly f_(k)=f_(k)(c)(n)(n);}}}
for (c=1; c<=d; c++)
{for (k=1; k<=size(J); k++)
{for (i=1; i<=n; i++)
{for (j=1; j<=n; j++){kill(f_(k)(c)(i)(j));}}}};};
int s = size(J);
ideal Jb = f_(1..s);
ideal J_b= groebner(Jb);
kill(Jb);
```

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