CROSS-SECTIONS OF DIVISIBLE ABELIAN o-GROUPS VIA TAME PAIRS

RICARDO PALOMINO PIEPENBORN

Abstract. In this note several equivalent characterizations are given for a divisible subgroup $\Delta' \subseteq \Gamma$ of a divisible group Γ to be the image of a section of a given surjective o-group homomorphism $f : \Gamma \twoheadrightarrow \Delta$ using the order-theoretic notion of tameness (equivalently, relative Dedekind completeness). The note concludes with an application of these characterizations to real closed valued fields.

Throughout this note all groups are abelian; moreover, an *o-group* is a totally ordered (abelian) group. If A is a ring, then $A^\times$ denotes its underlying group of multiplicative units, and if A is an o-group, then define $A^{>0} := \{a \in A \mid a > 0\}$.

1. Tame Pairs of Dense Linear Orders

Definition 1.1 (1.12 in [DL95], or [Pil94])**.** Let $(A, <) \subseteq (B, <)$ be an embedding of dense linear orders. Say that A *is tame in* B (or A *is Dedekind complete in* B) if for every A-bounded $b \in B$ (that is, for every $b \in \text{c.h.}_B(A) := \{b' \in B \mid \exists a_1, a_2 \in A \text{ such that } a_1 \leq b' \leq a_2\}$) there exists $a \in A$ such that one of the following items holds true:

(i) $b = a$, or
(ii) $b < a$ and there is no $a' \in A$ such that $b < a' < a$, or
(iii) $a < b$ and there is no $a' \in A$ such that $a < a' < b$.

Remark 1.2. Let $(A, <) \subseteq (B, <)$ be an embedding of dense linear orders. If A is tame in B, then it follows form the fact that $(A, <)$ is a dense linear order that for every A-bounded $b \in B$ (that is, for every $b \in \text{c.h.}_B(A)$) there exists a unique $a \in A$ such that exactly one of the items (i) - (iii) in Definition 1.1 holds for a and b.

Definition 1.3. Let $(A, <) \subseteq (B, <)$ be an embedding of dense linear orders and suppose that A is tame in B. The *standard part map associated with the tame pair* $A \subseteq B$ is the map $\text{st}_A^B : \text{c.h.}_B(A) \twoheadrightarrow A$ given by setting $\text{st}_A^B(b)$ ($b \in \text{c.h.}_B(A)$) to be the unique element in A for which one of the items (i) - (iii) in Definition 1.1 hold for $\text{st}_A^B(b)$ and b. If A and B are clear from the context, then write $\text{st} := \text{st}_A^B$.

Remark 1.4. Let $(A, <) \subseteq (B, <)$ be an embedding of dense linear orders and suppose that A is tame in B. Then:

(i) $\text{st}(a) = a$ for all $a \in A \subseteq \text{c.h.}_B(A)$.
(ii) If $b \in \text{c.h.}_B(A) \setminus A$ and $b < \text{st}(b)$, then for all $a \in A$ such that $a < b$ there exists $a' \in A$ such that $a < a' < b$.

Lemma 1.5. *Let $(A, \leq) \subseteq (B, \leq)$ be an embedding of dense linear orders. The following are equivalent:*

(i) *A is tame in B.*
(ii) *For every $b \in B$, the set $\{a \in A \mid a < b\}$ has a supremum in $A \cup \{\pm\infty\}$.*

Date: March 5, 2024.

Suppose further that A and B are o-groups and that A is a subgroup of B. Then (i) *and* (ii) *are equivalent to:*

(iii) For all $b \in \mathrm{c.h.}_B(A)$ there exists $a \in A$ such that $|b-a| < a'$ for all $a' \in A^{>0}$.

Proof. Straightforward from the definitions. □

1.1. **Tame pairs of o-minimal structures.** If $(A, <, \dots) \preceq (B, <, \dots)$ is an elementary extension of o-minimal structures ([Dri98]) and $D \subseteq A^n$ is an A-definable subset of A, then write D_B for the definable subset in B^n given by the same formula defining D in A^n. Moreover, say that $\bar{b} = (b_1, \dots, b_n) \in B^n$ is *A-bounded* if b_i is A-bounded for all $i \in \{1, \dots, n\}$, and if A is tame in B and $\bar{b} \in B^n$ is A-bounded, write $\mathrm{st}(\bar{b})$ for $(\mathrm{st}(b_1), \dots, \mathrm{st}(b_n))$.

Lemma 1.6. *Let $(A, <, \dots)$ be o-minimal and tame in an elementary extension $(B, <, \dots)$. Let $f : D \longrightarrow A$ be a continuous A-definable function on an A-definable set $D \subseteq A^n$, and let $\bar{b} \in D_B$ be A-bounded with $\mathrm{st}(\bar{b}) \in D$. Then $f_B(\bar{b})$ is A-bounded and $\mathrm{st}(f_B(\bar{b})) = f(\mathrm{st}(\bar{b}))$.*

Proof. See 1.13 in [DL95]. □

2. Cross-Sections of Divisible Abelian o-Groups via Tame Pairs

Definition 2.1 (pp. 48 & 49 in [Fuc70])**.** Let Γ_0 be a subgroup of a group $(\Gamma, +, 0)$.

(i) A subgroup $\Delta \subseteq \Gamma$ is Γ_0-*high* if Δ is a subgroup maximal for subset inclusion in Γ with $\Delta \cap \Gamma_0 = (0)$; in particular, $\Delta + \Gamma_0 = \Delta \oplus \Gamma_0$.
(ii) Γ_0 is an *absolute direct summand of* Γ if $\Gamma = \Delta \oplus \Gamma_0$ for every Γ_0-high subgroup $\Delta \subseteq \Gamma$.

Proposition 2.2. *Let Γ_0 be a divisible subgroup of a group $(\Gamma, +, 0)$. Then Γ_0 is an absolute direct summand on Γ.*

Proof. See [Fuc70, Theorem 21.2]. □

Corollary 2.3. *Let $f : \Gamma \twoheadrightarrow \Delta$ be a surjective group homomorphism and $\Delta' \subseteq \Gamma$ be a subgroup. Consider the following statements:*

(i) *The map $f_{\restriction \Delta'} : \Delta' \longrightarrow \Delta$ is a group isomorphism; in particular, $(f_{\restriction \Delta'})^{-1} : \Delta \hookrightarrow \Gamma$ is a section of $f : \Gamma \twoheadrightarrow \Delta$.*
(ii) *$\Gamma = \Delta' \oplus \ker(f)$.*
(iii) *Δ' is a subgroup maximal for subset inclusion in Γ with $\Delta' \cap \ker(f) = (0)$.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii)*, and if $\ker(f)$ is divisible, then all statements are equivalent.*

Proof. (i) $\Rightarrow$ (ii). Since $f_{\restriction \Delta'}$ is injective, $\Delta' \cap \ker(f) = 0$, hence $\Delta' + \ker(f) = \Delta' \oplus \ker(f)$. Pick $\gamma \in \Gamma$; since $f_{\restriction \Delta'}$ is surjective, there exists $\delta' \in \Delta'$ with $f(\delta') = f(\gamma)$, hence $\gamma - \delta' \in \ker(f)$ and thus $\gamma = \delta' + (\gamma - \delta') \in \Delta' \oplus \ker(f)$.

(ii) $\Rightarrow$ (i). Obvious.

(ii) $\Rightarrow$ (iii). Assume for contradiction that item (iii) does not hold and let $\Delta' \subsetneq \Gamma' \subseteq \Gamma$ be a subgroup maximal for subset inclusion in Γ with $\Gamma' \cap \ker(f) = (0)$. Pick $\gamma' \in \Gamma' \setminus \Delta'$; by assumption, there exist $\delta' \in \Delta'$ and $0 \neq \eta \in \ker(f)$ such that $\gamma' = \delta' + \eta$, therefore $0 \neq \eta = \gamma' - \delta' \in \Gamma' \cap \ker(f)$, a contradiction.

(iii) $\Rightarrow$ (ii). Δ' is $\ker(f)$-high by assumption; since $\ker(f)$ is divisible, it is an absolute direct summand of Γ by Proposition 2.2, hence $\Gamma = \Delta' \oplus \ker(f)$. □

Theorem 2.4. *Let* Γ *and* Δ *be divisible o-groups,* $f : \Gamma \twoheadrightarrow \Delta$ *be a surjective o-group homomorphism, and* $\Delta' \subseteq \Gamma$ *be a divisible subgroup (in particular,* $(\Delta', <)$ *is a dense linear order). The following are equivalent:*

(i) *The map* $f_{\restriction \Delta'} : \Delta' \longrightarrow \Delta$ *is an o-group isomorphism; in particular,* $(f_{\restriction \Delta'})^{-1} : \Delta \hookrightarrow \Gamma$ *is a section of the o-group homomorphism* $f : \Gamma \twoheadrightarrow \Delta$.
(ii) $\Gamma = \Delta' \oplus \ker(f)$.
(iii) Δ' *is a subgroup maximal for subset inclusion in* Γ *with* $\Delta' \cap \ker(f) = (0)$.
(iv) Δ' *is tame and cofinal in* Γ*, and* $\ker(f) = \{\gamma \in \Gamma \mid \mathrm{st}(\gamma) = 0\}$*, where* $\mathrm{st} : \Gamma \twoheadrightarrow \Delta'$ *is the standard part map associated with the tame pair* $\Delta' \subseteq \Gamma$.
(v) Δ' *is tame and cofinal in* Γ*, and* $f(\gamma) \geq 0$ *if and only if* $\mathrm{st}(\gamma) \geq 0$ *for all* $\gamma \in \Gamma$*, where* $\mathrm{st} : \Gamma \twoheadrightarrow \Delta'$ *is the standard part map associated with the tame pair* $\Delta' \subseteq \Gamma$.

In particular, if any of the items (i) - (v) *hold, then:*

- Δ' *is tame and cofinal in* Γ,
- $\mathrm{st}(\gamma)$ *is the unique element in* Δ' *such that* $f(\mathrm{st}(\gamma)) = f(\gamma)$ *for all* $\gamma \in \Gamma$*, and*
- *the standard part map* $\mathrm{st} : \Gamma \twoheadrightarrow \Delta'$ *associated with the tame pair* $\Delta' \subseteq \Gamma$ *is a retract of* $\Delta' \subseteq \Gamma$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). Since f is a surjective o-group homomorphism, $\ker(f)$ is convex in Γ, and since convex subgroups of divisible o-groups are divisible, the equivalence of items (i) - (iii) follows from Corollary 2.3.

(i) $\Rightarrow$ (iv). To prove that Δ' is cofinal in Γ, pick $\gamma \in \Gamma$ with $0 < \gamma$. Since $(\Delta, <)$ has no end points, there exists $\delta \in \Delta$ with $f(\gamma) < \delta$, and since $f_{\restriction \Delta'}$ is surjective, there exists $\delta' \in \Delta'$ such that $f(\delta') = \delta$; but then $\gamma < \delta'$, as otherwise $\delta' \leq \gamma$ would imply that $\delta = f(\delta') \leq f(\gamma)$, hence Δ' is cofinal in Γ and thus every $\gamma \in \Gamma$ is Δ'-bounded. To prove that Δ' is tame in Γ, pick any $\gamma \in \Gamma$ and assume without loss of generality that $0 < \gamma$ (otherwise replace γ by $-\gamma$). Since $f_{\restriction \Delta'}$ is bijective by assumption, there exists a unique $\delta' \in \Delta'$ such that $f(\gamma) = f(\delta')$, i.e., $\delta' - \gamma \in \ker(f)$. Note that $0 \leq \delta'$; otherwise, $\delta' < 0$ implies that $f(\gamma) = f(\delta') < f(0)$ since f is order-preserving and $f_{\restriction \Delta'}$ is injective, and $0 < \gamma$ implies $f(0) \leq f(\gamma)$, giving the required contradiction. It is now claimed that $\mathrm{st}(\gamma) = \delta'$. If $\gamma \in \Delta'$, then $\gamma = \delta'$ by choice of $\delta' \in \Delta'$ and thus $\mathrm{st}(\gamma) = \mathrm{st}(\delta') = \delta'$; if $\gamma \notin \Delta'$, then there are two possible cases:

- Case 1: $\gamma < \delta'$. Assume for contradiction that there exists $\delta'_1 \in \Delta'$ such that $\gamma < \delta'_1 < \delta'$. Then $0 < \delta'_1 - \gamma < \delta' - \gamma$, and since $\ker(f)$ is convex in Γ and $\delta' - \gamma \in \ker(f)$, it follows that $\delta'_1 - \gamma \in \ker(f)$, hence $f(\delta'_1) = f(\gamma) = f(\delta')$, contradicting uniqueness of $\delta' \in \Delta'$.
- Case 2: $\delta' < \gamma$. Assume for contradiction that there exists $\delta'_1 \in \Delta'$ such that $\delta' < \delta'_1 < \gamma$; then $\delta' - \gamma < \delta'_1 - \gamma < 0$, and since $\ker(f)$ is convex in Γ and $\delta' - \gamma \in \ker(f)$, it follows that $\delta'_1 - \gamma \in \ker(f)$, hence $f(\delta'_1) = f(\gamma) = f(\delta')$, contradicting uniqueness of $\delta' \in \Delta'$.

Therefore Δ' is tame in Γ; in particular, this shows that for every $\gamma \in \Gamma$, $\mathrm{st}(\gamma)$ is the unique element in Δ' such that $f(\gamma) = f(\mathrm{st}(\gamma))$, i.e., $\mathrm{st}(\gamma)$ is the unique element in Δ' such that $\eta_\gamma := \gamma - \mathrm{st}(\gamma) \in \ker(f)$, hence

$$f(\gamma) = 0 \iff f(\mathrm{st}(\gamma) + \eta_\gamma) = 0 \iff f(\mathrm{st}(\gamma)) = 0 \iff \mathrm{st}(\gamma) = 0,$$

where the last equivalence follows from the assumption that $f_{\restriction \Delta'}$ is injective.

(iv) $\Rightarrow$ (i). $\ker(f) = \{\gamma \in \Gamma \mid \mathrm{st}(\gamma) = 0\}$ implies that $\Delta' \cap \ker(f) = (0)$, and thus $f_{\restriction \Delta'}$ is injective. To show that $f_{\restriction \Delta'}$ is surjective it suffices to prove that $f(\gamma) = f(\mathrm{st}(\gamma))$ for all $\gamma \in \Gamma$ (note that since Δ' is cofinal in Γ, $\mathrm{st}(\gamma)$ exists for all $\gamma \in \Gamma$); since $\ker(f) = \{\gamma \in \Gamma \mid \mathrm{st}(\gamma) = 0\}$, it suffices in turn to show that $\mathrm{st}(\gamma - \mathrm{st}(\gamma)) = 0$ for all $\gamma \in \Gamma$. If $\gamma \in \Delta'$, then $\gamma = \mathrm{st}(\gamma)$ and thus $f(\gamma) = f(\mathrm{st}(\gamma))$. Let now $\gamma \in \Gamma \setminus \Delta'$, assume without loss of generality that $0 < \gamma$ (otherwise replace γ by $-\gamma$), and assume for contradiction that $\mathrm{st}(\gamma - \mathrm{st}(\gamma)) \neq 0$.

- Case 1: $0 < \gamma < \mathrm{st}(\gamma)$. Then $\gamma - \mathrm{st}(\gamma) < 0$, and there are 2 possible subcases:

  - Subcase 1.1: $\mathrm{st}(\gamma - \mathrm{st}(\gamma)) < \gamma - \mathrm{st}(\gamma) < 0$. In this case, there must exist $\delta' \in \Delta'$ such that $\gamma - \mathrm{st}(\gamma) < \delta' < 0$, hence $\gamma < \delta + \mathrm{st}(\gamma) < \mathrm{st}(\gamma)$ and $\delta' + \mathrm{st}(\gamma) \in \Delta'$ is a contradiction to tameness of Δ' in Γ.
  - Subcase 1.2: $\gamma - \mathrm{st}(\gamma) < \mathrm{st}(\gamma - \mathrm{st}(\gamma)) < 0$. In this case, $\gamma < \mathrm{st}(\gamma) + \mathrm{st}(\gamma - \mathrm{st}(\gamma)) < \mathrm{st}(\gamma)$ and $\mathrm{st}(\gamma) + \mathrm{st}(\gamma - \mathrm{st}(\gamma)) \in \Delta'$ is a contradiction to tameness of Δ' in Γ.
- Case 2: $0 \leq \mathrm{st}(\gamma) < \gamma$. Then $0 < \gamma - \mathrm{st}(\gamma)$ and there are 2 possible subcases:
  - Subcase 2.1: $0 < \mathrm{st}(\gamma - \mathrm{st}(\gamma)) < \gamma - \mathrm{st}(\gamma)$. In this case $\mathrm{st}(\gamma) < \mathrm{st}(\gamma) + \mathrm{st}(\gamma - \mathrm{st}(\gamma)) < \gamma$ and $\mathrm{st}(\gamma) + \mathrm{st}(\gamma - \mathrm{st}(\gamma)) \in \Delta'$ is a contradiction to tameness of Δ' in Γ.
  - Subcase 2.2: $0 < \gamma - \mathrm{st}(\gamma) < \mathrm{st}(\gamma - \mathrm{st}(\gamma))$. In this case, there must exist $\delta' \in \Delta'$ such that $0 < \delta' < \gamma - \mathrm{st}(\gamma)$, hence $\mathrm{st}(\gamma) < \delta' + \mathrm{st}(\gamma) < \gamma$ and $\delta' + \mathrm{st}(\gamma) \in \Delta'$ is a contradiction to tameness of Δ' in Γ.

In each of the cases above a contradiction is reached, hence $\mathrm{st}(\gamma - \mathrm{st}(\gamma)) = 0$ for all $\gamma \in \Gamma$, i.e., $f(\gamma) = f(\mathrm{st}(\gamma))$ for all $\gamma \in \Gamma$, and thus $f_{\upharpoonright \Delta'} : \Delta' \longrightarrow \Gamma$ is surjective, as required.

(iv) $\Leftrightarrow$ (v). One direction is clear, so suppose that item (iv) holds, i.e., $f(\gamma) = 0$ if and only if $\mathrm{st}(\gamma) = 0$ for all $\gamma \in \Gamma$; it therefore suffices to show that $f(\gamma) > 0$ if and only if $\mathrm{st}(\gamma) > 0$ for all $\gamma \in \Gamma$. Pick $\gamma \in \Gamma$.

- Assume for contradiction that $f(\gamma) > 0$ and $\mathrm{st}(\gamma) \leq 0$. Since $\mathrm{st}(\gamma) = 0$ implies $f(\gamma) = 0$, it must be the case that $\mathrm{st}(\gamma) < 0$, and thus $\gamma \leq 0$, as otherwise $\mathrm{st}(\gamma) < 0 < \gamma$ contradicts tameness of Δ' in Γ. On the other hand, $0 < f(\gamma)$ implies that $0 \leq \gamma$, as otherwise $\gamma < 0$ implies $f(\gamma) \leq 0$; therefore $\gamma = 0$ and thus $f(\gamma) = f(0) > 0$, a contradiction.
- Assume for contradiction that $\mathrm{st}(\gamma) > 0$ and $f(\gamma) \leq 0$. Since $f(\gamma) = 0$ implies $\mathrm{st}(\gamma) = 0$, it must be the case that $f(\gamma) < 0$, and thus $\gamma \leq 0$, as otherwise $0 < \gamma$ implies $0 \leq f(\gamma)$. On the other hand, $\mathrm{st}(\gamma) > 0$ implies that $\gamma \geq 0$, as otherwise $\gamma < 0 < \mathrm{st}(\gamma)$ contradicts tameness of Δ' in Γ; therefore $\gamma = 0$ and thus $\mathrm{st}(\gamma) = \mathrm{st}(0) > 0$, a contradiction.

To conclude, suppose that any of the items (i) - (v) hold, so that Δ' is tame and cofinal in Γ, and $f_{\upharpoonright \Delta'} : \Delta' \longrightarrow \Delta$ is an o-group isomorphism; then it follows from the proof of the implication (i) $\Rightarrow$ (iv) that $\mathrm{st}(\gamma)$ is the unique element in Δ' such that $f(\mathrm{st}(\gamma)) = f(\gamma)$ for all $\gamma \in \Gamma$, hence $\mathrm{st}(\gamma) = (f_{\upharpoonright \Delta'})^{-1}(f(\gamma))$ for all $\gamma \in \Gamma$ and thus $\mathrm{st} = (f_{\upharpoonright \Delta'})^{-1} \circ f$ is a surjective o-group homomorphism such that $\mathrm{st}_{\upharpoonright \Delta'} = \mathrm{id}_{\Delta'}$, therefore $\mathrm{st} : \Gamma \longrightarrow \Delta'$ is a retract of $\Delta' \subseteq \Gamma$. □

Proposition 2.5. *Let $(\Gamma, +)$ and $(\Delta, +)$ be divisible o-groups (in particular, $(\Delta, <)$ is a dense linear order) such that $\Delta \subseteq \Gamma$. Suppose that Δ is tame and cofinal in Γ, and let $\mathrm{st} : \Gamma \twoheadrightarrow \Delta$ be the standard part map associated with the tame pair $\Delta \subseteq \Gamma$. Then $\mathrm{st} : \Gamma \twoheadrightarrow \Delta$ is a surjective o-group homomorphism; in particular, $\mathrm{st} : \Gamma \twoheadrightarrow \Delta$ is a retract of $\Delta \subseteq \Gamma$.*

Proof. If $\gamma \in \Gamma$ is such that $\gamma \geq 0$, then $\mathrm{st}(\gamma) \geq 0$, as otherwise $\mathrm{st}(\gamma) < 0 \leq \gamma$ contradicts tameness of Δ in Γ, hence $\mathrm{st} : \Gamma \twoheadrightarrow \Delta$ is order-preserving. Since the $\mathscr{L}^{\mathrm{og}} := \{+, -, 0, \leq\}$-theory of divisible o-groups is model complete and o-minimal, $\Delta \subseteq \Gamma$ is an elementary extension of o-minimal $\mathscr{L}^{\mathrm{og}}$-structures; since Δ and Γ are topological groups with respect to the order topology (i.e., $+$ and $-$ are continuous functions) it follows from Lemma 1.6 that $\mathrm{st}(\gamma_1 + \gamma_2) = \mathrm{st}(\gamma_1) + \mathrm{st}(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$ (here cofinality of Δ in Γ is deployed), therefore $\mathrm{st} : \Gamma \twoheadrightarrow \Delta$ is a surjective o-group homomorphism such that $\mathrm{st}_{\upharpoonright \Delta} = \mathrm{id}_{\Delta}$, hence it is a retract of $\Delta \subseteq \Gamma$. □

3. An Application to Real Closed Valued Fields

Recall that a *real closed valued field* is a valued field (K, v) (see [EP05] or [ADH17, Section 3]) such that K is a real closed field and v is a *convex valuation* (also called *order-compatible valuation*) on K, that is, $0 < a < b$ implies $v(b) \leq v(a)$ for all $a, b \in K$; equivalently, a real closed valued field is a pair (K, V) where K is a real closed field and V is a convex subring. If (K, v) is a real closed valued field, then write $V_v := \{a \in K \mid v(a) \geq 0\}$ for its corresponding valuation ring.

Corollary 3.1. *Let (K, v) be a real closed valued field with value group Γ and $G \subseteq K^{>0}$ be a divisible subgroup (in particular, $(G, <)$ is a dense linear order). The following are equivalent:*

(i) *G is a* monomial group of (K, v)*, that is, the map $v_{\restriction G} : G \longrightarrow \Gamma$ is a group isomorphism; in particular, $(v_{\restriction G})^{-1} : \Gamma \hookrightarrow K^{>0}$ is a section of the group homomorphism $v_{\restriction K^{>0}} : K^{>0} \twoheadrightarrow \Gamma$.*
(ii) *$K^{>0} = G \cdot \ker(v_{\restriction K^{>0}})$.*
(iii) *G is a subgroup maximal for subset inclusion in $K^{>0}$ with $G \cap \ker(v_{\restriction K^{>0}}) = (1)$.*
(iv) *G is tame and cofinal in $K^{>0}$, and $\ker(v_{\restriction K^{>0}}) = \{r \in K^{>0} \mid \mathrm{st}(r) = 1\}$, where $\mathrm{st} : K^{>0} \longrightarrow G$ is the standard part map associated with the tame pair $G \subseteq K^{>0}$.*
(v) *G is tame and cofinal in $K^{>0}$, and $v(r) \geq 0$ if and only if $\mathrm{st}(r) \leq 1$ for all $r \in K^{>0}$, where $\mathrm{st} : K^{>0} \longrightarrow G$ is the standard part map associated with the tame pair $G \subseteq K^{>0}$.*
(vi) *G is tame and cofinal in $K^{>0}$, and $V_v = \{a \in K \mid a = 0 \text{ or } \mathrm{st}(|a|) \leq 1\}$, where $\mathrm{st} : K^{>0} \longrightarrow G$ is the standard part map associated with the tame pair $G \subseteq K^{>0}$.*

In particular, if any of the items (i) - (vi) *hold, then:*

- *G is tame and cofinal in $K^{>0}$,*
- *$\mathrm{st}(r)$ is the unique element in G such that $v(\mathrm{st}(r)) = v(r)$ for all $r \in K^{>0}$, and*
- *the standard part map $\mathrm{st} : K^{>0} \twoheadrightarrow G$ associated with the tame pair $G \subseteq K^{>0}$ is a retract of $G \subseteq K^{>0}$.*

Proof. Since (K, v) is a real closed valued field, $K^{>0}$ and Γ are divisible o-groups and the composite map $(-) \circ v_{\restriction K^{>0}} : K^{>0} \twoheadrightarrow \Gamma \twoheadrightarrow \Gamma^{\mathrm{op}}$ is a surjective o-group homomorphism such that $\ker((-) \circ v_{\restriction K^{>0}}) = \ker(v_{\restriction K^{>0}})$, and thus the equivalence of items (i) - (v) follows from Theorem 2.4; moreover, the equivalence of items (v) and (vi) is clear since $V_v = \{a \in K \mid v(a) \geq 0\}$ and $v(a) = v(-a)$ for all $a \in K$. □

Example 3.2. Let K be a real closed field and Γ be a divisible o-group. Then the field of Hahn series $K((\Gamma)) := K((x^{\Gamma}))$ is a real closed valued field with value group Γ and $x^{\Gamma} := \{x^{\gamma} \mid \gamma \in \Gamma\}$ is a divisible subgroup of $K((\Gamma))^{>0}$ such that $v_{\restriction x^{\Gamma}} : x^{\Gamma} \longrightarrow \Gamma$ is a group isomorphism; therefore x^{Γ} is tame in $K((\Gamma))^{>0}$ and $\mathrm{st}(r) = x^{v(r)}$ for all $r \in K((\Gamma))^{>0}$ by Corollary 3.1.

Given a real closed valued field K, one can therefore identify the monomial groups of K with certain tame and cofinal divisible subgroups of $K^{>0}$ by Corollary 3.1. Conversely, order-compatible valuations on K are induced by certain tame and cofinal divisible subgroups of $K^{>0}$:

Lemma 3.3. *Let K be a real closed field and $G \subseteq K^{>0}$ be a tame and cofinal divisible subgroup. The following are equivalent:*

(i) *The map $v_G : K^{\times} \twoheadrightarrow G^{\mathrm{op}}$ given by $v_G(a) := \mathrm{st}(|a|)$ is an order-compatible valuation on K; in particular, G is a monomial group of the real closed valued field (K, v_G), and the corresponding convex valuation ring is $V_G := \{0\} \cup \{a \in K^{\times} \mid \mathrm{st}(|a|) \leq 1\}$.*
(ii) $\mathrm{st}(2) = 1$
(iii) $\mathrm{st}(2) \leq 1$.

Proof. (i) ⇒ (ii). Since v_G is a valuation on K, the group of units of its corresponding valuation ring V_G is $V_G^{\times} = \{a \in K^{\times} \mid v_G(a) = 1\} = \{a \in K^{\times} \mid \mathrm{st}(|a|) = 1\}$, and since $2 \in V_G$, (ii) follows.

(ii) ⇒ (iii). Clear.

(iii) $\Rightarrow$ (i). By choice of K and G, it follows from Proposition 2.5 that the standard part map $\mathrm{st} : K^{>0} \longrightarrow\!\!\!\!\rightarrow G$ is a surjective morphism of o-groups, and thus $v_G : (K^\times, \cdot) \longrightarrow (G^{\mathrm{op}}, \cdot)$ is a surjective group homomorphism such that $a < b$ in $K^{>0}$ implies $v_G(b) \leq v_G(a)$ in G^{op}, so it remains to show that for all $a, b \in K^\times$ with $a \neq -b$, $v_G(a+b) \geq \min\{v_G(a), v_G(b)\}$ in G^{op}, i.e., $\mathrm{st}(|a+b|) \leq \max\{\mathrm{st}(|a|), \mathrm{st}(|b|)\}$ in G. Pick $a, b \in K^\times$ with $a \neq -b$ and assume without loss of generality that $|a| \leq |b|$, so that $\max\{\mathrm{st}(|a|), \mathrm{st}(|b|)\} = \mathrm{st}(|b|)$; then $|a+b| \leq |a|+|b| \leq 2|b|$, therefore $\mathrm{st}(|a + b|) \leq \mathrm{st}(2)\mathrm{st}(|b|) \leq \mathrm{st}(|b|)$, as required. □

References

[ADH17] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven. *Asymptotic differential algebra and model theory of transseries*. Vol. 195. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2017, pp. xxi+849.

[DL95] Lou van den Dries and Adam H. Lewenberg. "T-convexity and tame extensions". In: *J. Symbolic Logic* 60.1 (1995), pp. 74–102.

[Dri98] Lou van den Dries. *Tame topology and o-minimal structures*. Vol. 248. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1998, pp. x+180.

[EP05] Antonio J. Engler and Alexander Prestel. *Valued fields*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005, pp. x+205.

[Fuc70] L. Fuchs. *Infinite Abelian Groups. Volume 1*. Academic Press, 1970.

[Pil94] Anand Pillay. "Definability of types, and pairs of O-minimal structures". In: *J. Symbolic Logic* 59.4 (1994), pp. 1400–1409.

The University of Manchester, Department of Mathematics, Oxford Road, Manchester, M13 9PL, United Kingdom

Email address: ricardo.palomino@rjpp.net

URL: rjpp.net